

Integer Set Library: Manual

Version: isl-0.14.1

Sven Verdoolaege

April 9, 2015

Contents

1	User Manual	3
1.1	Introduction	3
1.1.1	Backward Incompatible Changes	3
1.2	License	6
1.3	Installation	6
1.3.1	Installation from the git repository	7
1.3.2	Common installation instructions	7
1.4	Integer Set Library	8
1.4.1	Initialization	8
1.4.2	Memory Management	9
1.4.3	Values	9
1.4.4	Sets and Relations	13
1.4.5	Error Handling	14
1.4.6	Identifiers	14
1.4.7	Spaces	15
1.4.8	Local Spaces	20
1.4.9	Input and Output	22
1.4.10	Creating New Sets and Relations	25
1.4.11	Inspecting Sets and Relations	31
1.4.12	Properties	39
1.4.13	Unary Operations	45
1.4.14	Binary Operations	61
1.4.15	Lists	72
1.4.16	Associative arrays	73
1.4.17	Multiple Values	75
1.4.18	Vectors	77
1.4.19	Matrices	78
1.4.20	Piecewise Quasi Affine Expressions	79
1.4.21	Piecewise Multiple Quasi Affine Expressions	89
1.4.22	Points	102
1.4.23	Piecewise Quasipolynomials	103
1.4.24	Bounds on Piecewise Quasipolynomials and Piecewise Quasipolynomial Reductions	110
1.4.25	Parametric Vertex Enumeration	115

1.5	Polyhedral Compilation Library	116
1.5.1	Dependence Analysis	116
1.5.2	Scheduling	120
1.5.3	AST Generation	126
1.6	Applications	143
1.6.1	<code>isl_polyhedron_sample</code>	144
1.6.2	<code>isl_pip</code>	144
1.6.3	<code>isl_polyhedron_minimize</code>	144
1.6.4	<code>isl_polytope_scan</code>	144
1.6.5	<code>isl_codegen</code>	144
2	Implementation Details	145
2.1	Sets and Relations	145
2.2	Simple Hull	146
2.3	Parametric Integer Programming	147
2.3.1	Introduction	147
2.3.2	The Dual Simplex Method	147
2.3.3	Gomory Cuts	148
2.3.4	Negative Unknowns and Maximization	149
2.3.5	Preprocessing	150
2.3.6	Postprocessing	151
2.3.7	Context Tableau	152
2.3.8	Experiments	154
2.3.9	Online Symmetry Detection	155
2.4	Coalescing	156
2.5	Transitive Closure	157
2.5.1	Introduction	157
2.5.2	Computing an Approximation of R^k	157
2.5.3	Checking Exactness	163
2.5.4	Decomposing R into strongly connected components	164
2.5.5	Partitioning the domains and ranges of R	166
2.5.6	Incremental Computation	168
2.5.7	An Omega-like implementation	170
	References	171

Chapter 1

User Manual

1.1 Introduction

`isl` is a thread-safe C library for manipulating sets and relations of integer points bounded by affine constraints. The descriptions of the sets and relations may involve both parameters and existentially quantified variables. All computations are performed in exact integer arithmetic using `GMP` or `imath`. The `isl` library offers functionality that is similar to that offered by the `Omega` and `Omega+` libraries, but the underlying algorithms are in most cases completely different.

The library is by no means complete and some fairly basic functionality is still missing. Still, even in its current form, the library has been successfully used as a backend polyhedral library for the polyhedral scanner `CLoG` and as part of an equivalence checker of static affine programs. For bug reports, feature requests and questions, visit the discussion group at <http://groups.google.com/group/isl-development>.

1.1.1 Backward Incompatible Changes

Changes since `isl-0.02`

- The old printing functions have been deprecated and replaced by `isl_printer` functions, see [Input and Output](#).
- Most functions related to dependence analysis have acquired an extra `must` argument. To obtain the old behavior, this argument should be given the value 1. See [Dependence Analysis](#).

Changes since `isl-0.03`

- The function `isl_pw_qpolynomial_fold_add` has been renamed to `isl_pw_qpolynomial_fold_fold`. Similarly, `isl_union_pw_qpolynomial_fold_add` has been renamed to `isl_union_pw_qpolynomial_fold_fold`.

Changes since isl-0.04

- All header files have been renamed from `isl_header.h` to `isl/header.h`.

Changes since isl-0.05

- The functions `isl_printer_print_basic_set` and `isl_printer_print_basic_map` no longer print a newline.
- The functions `isl_flow_get_no_source` and `isl_union_map_compute_flow` now return the accesses for which no source could be found instead of the iterations where those accesses occur.
- The functions `isl_basic_map_identity` and `isl_map_identity` now take a **map** space as input. An old call `isl_map_identity(space)` can be rewritten to `isl_map_identity(isl_space_map_from_set(space))`.
- The function `isl_map_power` no longer takes a parameter position as input. Instead, the exponent is now expressed as the domain of the resulting relation.

Changes since isl-0.06

- The format of `isl_printer_print_qpolynomial`'s `ISL_FORMAT_ISL` output has changed. Use `ISL_FORMAT_C` to obtain the old output.
- The `*_fast_*` functions have been renamed to `*_plain_*`. Some of the old names have been kept for backward compatibility, but they will be removed in the future.

Changes since isl-0.07

- The function `isl_pw_aff_max` has been renamed to `isl_pw_aff_union_max`. Similarly, the function `isl_pw_aff_add` has been renamed to `isl_pw_aff_union_add`.
- The `isl_dim` type has been renamed to `isl_space` along with the associated functions. Some of the old names have been kept for backward compatibility, but they will be removed in the future.
- Spaces of maps, sets and parameter domains are now treated differently. The distinction between map spaces and set spaces has always been made on a conceptual level, but proper use of such spaces was never checked. Furthermore, up until isl-0.07 there was no way of explicitly creating a parameter space. These can now be created directly using `isl_space_params_alloc` or from other spaces using `isl_space_params`.
- The space in which `isl_aff`, `isl_pw_aff`, `isl_qpolynomial`, `isl_pw_qpolynomial`, `isl_qpolynomial_fold` and `isl_pw_qpolynomial_fold` objects live is now a map space instead of a set space. This means, for example, that the dimensions

of the domain of an `isl_aff` are now considered to be of type `isl_dim_in` instead of `isl_dim_set`. Extra functions have been added to obtain the domain space. Some of the constructors still take a domain space and have therefore been renamed.

- The functions `isl_equality_alloc` and `isl_inequality_alloc` now take an `isl_local_space` instead of an `isl_space`. An `isl_local_space` can be created from an `isl_space` using `isl_local_space_from_space`.
- The `isl_div` type has been removed. Functions that used to return an `isl_div` now return an `isl_aff`. Note that the space of an `isl_aff` is that of relation. When replacing a call to `isl_div_get_coefficient` by a call to `isl_aff_get_coefficient` any `isl_dim_set` argument needs to be replaced by `isl_dim_in`. A call to `isl_aff_from_div` can be replaced by a call to `isl_aff_floor`. A call to `isl_qpolynomial_div(div)` call be replaced by the nested call

```
isl_qpolynomial_from_aff(isl_aff_floor(div))
```

The function `isl_constraint_div` has also been renamed to `isl_constraint_get_div`.

- The `nparam` argument has been removed from `isl_map_read_from_str` and similar functions. When reading input in the original PolyLib format, the result will have no parameters. If parameters are expected, the caller may want to perform dimension manipulation on the result.

Changes since isl-0.09

- The `schedule_split_parallel` option has been replaced by the `schedule_split_scaled` option.
- The first argument of `isl_pw_aff_cond` is now an `isl_pw_aff` instead of an `isl_set`. A call `isl_pw_aff_cond(a, b, c)` can be replaced by

```
isl_pw_aff_cond(isl_set_indicator_function(a), b, c)
```

Changes since isl-0.10

- The functions `isl_set_dim_has_lower_bound` and `isl_set_dim_has_upper_bound` have been renamed to `isl_set_dim_has_any_lower_bound` and `isl_set_dim_has_any_upper_bound`. The new `isl_set_dim_has_lower_bound` and `isl_set_dim_has_upper_bound` have slightly different meanings.

Changes since isl-0.12

- `isl_int` has been replaced by `isl_val`. Some of the old functions are still available in `isl/deprecated/*.h` but they will be removed in the future.

- The functions `isl_pw_qpolynomial_eval`, `isl_union_pw_qpolynomial_eval`, `isl_pw_qpolynomial_fold_eval` and `isl_union_pw_qpolynomial_fold_eval` have been changed to return an `isl_val` instead of an `isl_qpolynomial`.
- The function `isl_band_member_is_zero_distance` has been removed. Essentially the same functionality is available through `isl_band_member_is_coincident`, except that it requires setting up coincidence constraints. The option `schedule_outer_zero_distance` has accordingly been replaced by the option `schedule_outer_coincidence`.
- The function `isl_vertex_get_expr` has been changed to return an `isl_multi_aff` instead of a rational `isl_basic_set`. The function `isl_vertex_get_domain` has been changed to return a regular basic set, rather than a rational basic set.

1.2 License

`isl` is released under the MIT license.

Permission is hereby granted, free of charge, to any person obtaining a copy of this software and associated documentation files (the "Software"), to deal in the Software without restriction, including without limitation the rights to use, copy, modify, merge, publish, distribute, sublicense, and/or sell copies of the Software, and to permit persons to whom the Software is furnished to do so, subject to the following conditions:

The above copyright notice and this permission notice shall be included in all copies or substantial portions of the Software.

THE SOFTWARE IS PROVIDED "AS IS", WITHOUT WARRANTY OF ANY KIND, EXPRESS OR IMPLIED, INCLUDING BUT NOT LIMITED TO THE WARRANTIES OF MERCHANTABILITY, FITNESS FOR A PARTICULAR PURPOSE AND NONINFRINGEMENT. IN NO EVENT SHALL THE AUTHORS OR COPYRIGHT HOLDERS BE LIABLE FOR ANY CLAIM, DAMAGES OR OTHER LIABILITY, WHETHER IN AN ACTION OF CONTRACT, TORT OR OTHERWISE, ARISING FROM, OUT OF OR IN CONNECTION WITH THE SOFTWARE OR THE USE OR OTHER DEALINGS IN THE SOFTWARE.

Note that by default `isl` requires `GMP`, which is released under the GNU Lesser General Public License (LGPL). This means that code linked against `isl` is also linked against LGPL code.

When configuring with `--with-int=imath`, `isl` will link against `imath`, a library for exact integer arithmetic released under the MIT license.

1.3 Installation

The source of `isl` can be obtained either as a tarball or from the git repository. Both are available from <http://freshmeat.net/projects/isl/>. The installation process depends on how you obtained the source.

1.3.1 Installation from the git repository

1. Clone or update the repository

The first time the source is obtained, you need to clone the repository.

```
git clone git://repo.or.cz/isl.git
```

To obtain updates, you need to pull in the latest changes

```
git pull
```

2. Optionally get `imath` submodule

To build `isl` with `imath`, you need to obtain the `imath` submodule by running in the git source tree of `isl`

```
git submodule init
git submodule update
```

This will fetch the required version of `imath` in a subdirectory of `isl`.

3. Generate configure

```
./autogen.sh
```

After performing the above steps, continue with the Common installation instructions.

1.3.2 Common installation instructions

1. Obtain GMP

By default, building `isl` requires GMP, including its headers files. Your distribution may not provide these header files by default and you may need to install a package called `gmp-devel` or something similar. Alternatively, GMP can be built from source, available from <http://gmplib.org/>. GMP is not needed if you build `isl` with `imath`.

2. Configure

`isl` uses the standard `autoconf` configure script. To run it, just type

```
./configure
```

optionally followed by some configure options. A complete list of options can be obtained by running

```
./configure --help
```


Below we discuss some of the more common options.

--prefix

Installation prefix for `isl`

--with-int=[gmp|imath]

Select the integer library to be used by `isl`, the default is `gmp`. Note that `isl` may run significantly slower if you use `imath`.

--with-gmp-prefix

Installation prefix for GMP (architecture-independent files).

--with-gmp-exec-prefix

Installation prefix for GMP (architecture-dependent files).

3. Compile

```
make
```

4. Install (optional)

```
make install
```

1.4 Integer Set Library

1.4.1 Initialization

All manipulations of integer sets and relations occur within the context of an `isl_ctx`. A given `isl_ctx` can only be used within a single thread. All arguments of a function are required to have been allocated within the same context. There are currently no functions available for moving an object from one `isl_ctx` to another `isl_ctx`. This means that there is currently no way of safely moving an object from one thread to another, unless the whole `isl_ctx` is moved.

An `isl_ctx` can be allocated using `isl_ctx_alloc` and freed using `isl_ctx_free`. All objects allocated within an `isl_ctx` should be freed before the `isl_ctx` itself is freed.

```
isl_ctx *isl_ctx_alloc();  
void isl_ctx_free(isl_ctx *ctx);
```

The user can impose a bound on the number of low-level *operations* that can be performed by an `isl_ctx`. This bound can be set and retrieved using the following functions. A bound of zero means that no bound is imposed. The number of operations performed can be reset using `isl_ctx_reset_operations`. Note that the number of low-level operations needed to perform a high-level computation may differ significantly across different versions of `isl`, but it should be the same across different platforms for the same version of `isl`.

Warning: This feature is experimental. `isl` has good support to abort and bail out during the computation, but this feature may exercise error code paths that are normally not used that much. Consequently, it is not unlikely that hidden bugs will be exposed.

```
void isl_ctx_set_max_operations(isl_ctx *ctx,
                               unsigned long max_operations);
unsigned long isl_ctx_get_max_operations(isl_ctx *ctx);
void isl_ctx_reset_operations(isl_ctx *ctx);
```

1.4.2 Memory Management

Since a high-level operation on isl objects usually involves several substeps and since the user is usually not interested in the intermediate results, most functions that return a new object will also release all the objects passed as arguments. If the user still wants to use one or more of these arguments after the function call, she should pass along a copy of the object rather than the object itself. The user is then responsible for making sure that the original object gets used somewhere else or is explicitly freed.

The arguments and return values of all documented functions are annotated to make clear which arguments are released and which arguments are preserved. In particular, the following annotations are used

`__isl_give`

`__isl_give` means that a new object is returned. The user should make sure that the returned pointer is used exactly once as a value for an `__isl_take` argument. In between, it can be used as a value for as many `__isl_keep` arguments as the user likes. There is one exception, and that is the case where the pointer returned is NULL. In this case, the user is free to use it as an `__isl_take` argument or not.

`__isl_null`

`__isl_null` means that a NULL value is returned.

`__isl_take`

`__isl_take` means that the object the argument points to is taken over by the function and may no longer be used by the user as an argument to any other function. The pointer value must be one returned by a function returning an `__isl_give` pointer. If the user passes in a NULL value, then this will be treated as an error in the sense that the function will not perform its usual operation. However, it will still make sure that all the other `__isl_take` arguments are released.

`__isl_keep`

`__isl_keep` means that the function will only use the object temporarily. After the function has finished, the user can still use it as an argument to other functions. A NULL value will be treated in the same way as a NULL value for an `__isl_take` argument.

1.4.3 Values

An `isl_val` represents an integer value, a rational value or one of three special values, infinity, negative infinity and NaN. Some predefined values can be created using the following functions.

```

#include <isl/val.h>
__isl_give isl_val *isl_val_zero(isl_ctx *ctx);
__isl_give isl_val *isl_val_one(isl_ctx *ctx);
__isl_give isl_val *isl_val_negone(isl_ctx *ctx);
__isl_give isl_val *isl_val_nan(isl_ctx *ctx);
__isl_give isl_val *isl_val_infty(isl_ctx *ctx);
__isl_give isl_val *isl_val_neginfty(isl_ctx *ctx);

```

Specific integer values can be created using the following functions.

```

#include <isl/val.h>
__isl_give isl_val *isl_val_int_from_si(isl_ctx *ctx,
    long i);
__isl_give isl_val *isl_val_int_from_ui(isl_ctx *ctx,
    unsigned long u);
__isl_give isl_val *isl_val_int_from_chunks(isl_ctx *ctx,
    size_t n, size_t size, const void *chunks);

```

The function `isl_val_int_from_chunks` constructs an `isl_val` from the n *digits*, each consisting of `size` bytes, stored at `chunks`. The least significant digit is assumed to be stored first.

Value objects can be copied and freed using the following functions.

```

#include <isl/val.h>
__isl_give isl_val *isl_val_copy(__isl_keep isl_val *v);
__isl_null isl_val *isl_val_free(__isl_take isl_val *v);

```

They can be inspected using the following functions.

```

#include <isl/val.h>
isl_ctx *isl_val_get_ctx(__isl_keep isl_val *val);
long isl_val_get_num_si(__isl_keep isl_val *v);
long isl_val_get_den_si(__isl_keep isl_val *v);
double isl_val_get_d(__isl_keep isl_val *v);
size_t isl_val_n_abs_num_chunks(__isl_keep isl_val *v,
    size_t size);
int isl_val_get_abs_num_chunks(__isl_keep isl_val *v,
    size_t size, void *chunks);

```

`isl_val_n_abs_num_chunks` returns the number of *digits* of `size` bytes needed to store the absolute value of the numerator of `v`. `isl_val_get_abs_num_chunks` stores these digits at `chunks`, which is assumed to have been preallocated by the caller. The least significant digit is stored first. Note that `isl_val_get_num_si`, `isl_val_get_den_si`, `isl_val_get_d`, `isl_val_n_abs_num_chunks` and `isl_val_get_abs_num_chunks` can only be applied to rational values.

An `isl_val` can be modified using the following function.

```
#include <isl/val.h>
__isl_give isl_val *isl_val_set_si(__isl_take isl_val *v,
    long i);
```

The following unary properties are defined on `isl_vals`.

```
#include <isl/val.h>
int isl_val_sgn(__isl_keep isl_val *v);
int isl_val_is_zero(__isl_keep isl_val *v);
int isl_val_is_one(__isl_keep isl_val *v);
int isl_val_is_negone(__isl_keep isl_val *v);
int isl_val_is_nonneg(__isl_keep isl_val *v);
int isl_val_is_nonpos(__isl_keep isl_val *v);
int isl_val_is_pos(__isl_keep isl_val *v);
int isl_val_is_neg(__isl_keep isl_val *v);
int isl_val_is_int(__isl_keep isl_val *v);
int isl_val_is_rat(__isl_keep isl_val *v);
int isl_val_is_nan(__isl_keep isl_val *v);
int isl_val_is_infty(__isl_keep isl_val *v);
int isl_val_is_neginfty(__isl_keep isl_val *v);
```

Note that the sign of NaN is undefined.

The following binary properties are defined on pairs of `isl_vals`.

```
#include <isl/val.h>
int isl_val_lt(__isl_keep isl_val *v1,
    __isl_keep isl_val *v2);
int isl_val_le(__isl_keep isl_val *v1,
    __isl_keep isl_val *v2);
int isl_val_gt(__isl_keep isl_val *v1,
    __isl_keep isl_val *v2);
int isl_val_ge(__isl_keep isl_val *v1,
    __isl_keep isl_val *v2);
int isl_val_eq(__isl_keep isl_val *v1,
    __isl_keep isl_val *v2);
int isl_val_ne(__isl_keep isl_val *v1,
    __isl_keep isl_val *v2);
```

For integer `isl_vals` we additionally have the following binary property.

```
#include <isl/val.h>
int isl_val_is_divisible_by(__isl_keep isl_val *v1,
    __isl_keep isl_val *v2);
```

An `isl_val` can also be compared to an integer using the following function. The result is undefined for NaN.

```
#include <isl/val.h>
int isl_val_cmp_si(__isl_keep isl_val *v, long i);
```

The following unary operations are available on `isl_vals`.

```
#include <isl/val.h>
__isl_give isl_val *isl_val_abs(__isl_take isl_val *v);
__isl_give isl_val *isl_val_neg(__isl_take isl_val *v);
__isl_give isl_val *isl_val_floor(__isl_take isl_val *v);
__isl_give isl_val *isl_val_ceil(__isl_take isl_val *v);
__isl_give isl_val *isl_val_trunc(__isl_take isl_val *v);
__isl_give isl_val *isl_val_2exp(__isl_take isl_val *v);
```

The following binary operations are available on `isl_vals`.

```
#include <isl/val.h>
__isl_give isl_val *isl_val_min(__isl_take isl_val *v1,
                                __isl_take isl_val *v2);
__isl_give isl_val *isl_val_max(__isl_take isl_val *v1,
                                __isl_take isl_val *v2);
__isl_give isl_val *isl_val_add(__isl_take isl_val *v1,
                                __isl_take isl_val *v2);
__isl_give isl_val *isl_val_add_ui(__isl_take isl_val *v1,
                                    unsigned long v2);
__isl_give isl_val *isl_val_sub(__isl_take isl_val *v1,
                                __isl_take isl_val *v2);
__isl_give isl_val *isl_val_sub_ui(__isl_take isl_val *v1,
                                    unsigned long v2);
__isl_give isl_val *isl_val_mul(__isl_take isl_val *v1,
                                __isl_take isl_val *v2);
__isl_give isl_val *isl_val_mul_ui(__isl_take isl_val *v1,
                                    unsigned long v2);
__isl_give isl_val *isl_val_div(__isl_take isl_val *v1,
                                __isl_take isl_val *v2);
```

On integer values, we additionally have the following operations.

```
#include <isl/val.h>
__isl_give isl_val *isl_val_2exp(__isl_take isl_val *v);
__isl_give isl_val *isl_val_mod(__isl_take isl_val *v1,
                                __isl_take isl_val *v2);
__isl_give isl_val *isl_val_gcd(__isl_take isl_val *v1,
                                __isl_take isl_val *v2);
__isl_give isl_val *isl_val_gcdext(__isl_take isl_val *v1,
                                   __isl_take isl_val *v2,
                                   __isl_give isl_val **x,
                                   __isl_give isl_val **y);
```

The function `isl_val_gcdext` returns the greatest common divisor g of $v1$ and $v2$ as well as two integers $*x$ and $*y$ such that $*x * v1 + *y * v2 = g$.

A value can be read from input using

```
#include <isl/val.h>
__isl_give isl_val *isl_val_read_from_str(isl_ctx *ctx,
    const char *str);
```

A value can be printed using

```
#include <isl/val.h>
__isl_give isl_printer *isl_printer_print_val(
    __isl_take isl_printer *p, __isl_keep isl_val *v);
```

GMP specific functions

These functions are only available if `isl` has been compiled with GMP support.

Specific integer and rational values can be created from GMP values using the following functions.

```
#include <isl/val_gmp.h>
__isl_give isl_val *isl_val_int_from_gmp(isl_ctx *ctx,
    mpz_t z);
__isl_give isl_val *isl_val_from_gmp(isl_ctx *ctx,
    const mpz_t n, const mpz_t d);
```

The numerator and denominator of a rational value can be extracted as GMP values using the following functions.

```
#include <isl/val_gmp.h>
int isl_val_get_num_gmp(__isl_keep isl_val *v, mpz_t z);
int isl_val_get_den_gmp(__isl_keep isl_val *v, mpz_t z);
```

1.4.4 Sets and Relations

`isl` uses six types of objects for representing sets and relations, `isl_basic_set`, `isl_basic_map`, `isl_set`, `isl_map`, `isl_union_set` and `isl_union_map`. `isl_basic_set` and `isl_basic_map` represent sets and relations that can be described as a conjunction of affine constraints, while `isl_set` and `isl_map` represent unions of `isl_basic_sets` and `isl_basic_maps`, respectively. However, all `isl_basic_sets` or `isl_basic_maps` in the union need to live in the same space. `isl_union_sets` and `isl_union_maps` represent unions of `isl_sets` or `isl_maps` in *different* spaces, where spaces are considered different if they have a different number of dimensions and/or different names (see §1.4.7). The difference between sets and relations (maps) is that sets have one set of variables, while relations have two sets of variables, input variables and output variables.

1.4.5 Error Handling

isl supports different ways to react in case a runtime error is triggered. Runtime errors arise, e.g., if a function such as `isl_map_intersect` is called with two maps that have incompatible spaces. There are three possible ways to react on error: to warn, to continue or to abort.

The default behavior is to warn. In this mode, `isl` prints a warning, stores the last error in the corresponding `isl_ctx` and the function in which the error was triggered returns `NULL`. An error does not corrupt internal state, such that `isl` can continue to be used. `isl` also provides functions to read the last error and to reset the memory that stores the last error. The last error is only stored for information purposes. Its presence does not change the behavior of `isl`. Hence, resetting an error is not required to continue to use `isl`, but only to observe new errors.

```
#include <isl/ctx.h>
enum isl_error isl_ctx_last_error(isl_ctx *ctx);
void isl_ctx_reset_error(isl_ctx *ctx);
```

Another option is to continue on error. This is similar to warn on error mode, except that `isl` does not print any warning. This allows a program to implement its own error reporting.

The last option is to directly abort the execution of the program from within the `isl` library. This makes it obviously impossible to recover from an error, but it allows to directly spot the error location. By aborting on error, debuggers break at the location the error occurred and can provide a stack trace. Other tools that automatically provide stack traces on abort or that do not want to continue execution after an error was triggered may also prefer to abort on error.

The on error behavior of `isl` can be specified by calling `isl_options_set_on_error` or by setting the command line option `--isl-on-error`. Valid arguments for the function call are `ISL_ON_ERROR_WARN`, `ISL_ON_ERROR_CONTINUE` and `ISL_ON_ERROR_ABORT`. The choices for the command line option are `warn`, `continue` and `abort`. It is also possible to query the current error mode.

```
#include <isl/options.h>
int isl_options_set_on_error(isl_ctx *ctx, int val);
int isl_options_get_on_error(isl_ctx *ctx);
```

1.4.6 Identifiers

Identifiers are used to identify both individual dimensions and tuples of dimensions. They consist of an optional name and an optional user pointer. The name and the user pointer cannot both be `NULL`, however. Identifiers with the same name but different pointer values are considered to be distinct. Similarly, identifiers with different names but the same pointer value are also considered to be distinct. Equal identifiers are represented using the same object. Pairs of identifiers can therefore be tested for equality using the `==` operator. Identifiers can be constructed, copied, freed, inspected and printed using the following functions.

```

#include <isl/id.h>
__isl_give isl_id *isl_id_alloc(isl_ctx *ctx,
    __isl_keep const char *name, void *user);
__isl_give isl_id *isl_id_set_free_user(
    __isl_take isl_id *id,
    __isl_give void (*free_user)(void *user));
__isl_give isl_id *isl_id_copy(isl_id *id);
__isl_null isl_id *isl_id_free(__isl_take isl_id *id);

isl_ctx *isl_id_get_ctx(__isl_keep isl_id *id);
void *isl_id_get_user(__isl_keep isl_id *id);
__isl_keep const char *isl_id_get_name(__isl_keep isl_id *id);

__isl_give isl_printer *isl_printer_print_id(
    __isl_take isl_printer *p, __isl_keep isl_id *id);

```

The callback set by `isl_id.set_free_user` is called on the user pointer when the last reference to the `isl_id` is freed. Note that `isl_id.get_name` returns a pointer to some internal data structure, so the result can only be used while the corresponding `isl_id` is alive.

1.4.7 Spaces

Whenever a new set, relation or similar object is created from scratch, the space in which it lives needs to be specified using an `isl_space`. Each space involves zero or more parameters and zero, one or two tuples of set or input/output dimensions. The parameters and dimensions are identified by an `isl_dim_type` and a position. The type `isl_dim_param` refers to parameters, the type `isl_dim_set` refers to set dimensions (for spaces with a single tuple of dimensions) and the types `isl_dim_in` and `isl_dim_out` refer to input and output dimensions (for spaces with two tuples of dimensions). Local spaces (see §1.4.8) also contain dimensions of type `isl_dim_div`. Note that parameters are only identified by their position within a given object. Across different objects, parameters are (usually) identified by their names or identifiers. Only unnamed parameters are identified by their positions across objects. The use of unnamed parameters is discouraged.

```

#include <isl/space.h>
__isl_give isl_space *isl_space_alloc(isl_ctx *ctx,
    unsigned nparam, unsigned n_in, unsigned n_out);
__isl_give isl_space *isl_space_params_alloc(isl_ctx *ctx,
    unsigned nparam);
__isl_give isl_space *isl_space_set_alloc(isl_ctx *ctx,
    unsigned nparam, unsigned dim);
__isl_give isl_space *isl_space_copy(__isl_keep isl_space *space);
__isl_null isl_space *isl_space_free(__isl_take isl_space *space);
unsigned isl_space_dim(__isl_keep isl_space *space,
    enum isl_dim_type type);

```


The space used for creating a parameter domain needs to be created using `isl_space_params_alloc`. For other sets, the space needs to be created using `isl_space_set_alloc`, while for a relation, the space needs to be created using `isl_space_alloc`. `isl_space_dim` can be used to find out the number of dimensions of each type in a space, where type may be `isl_dim_param`, `isl_dim_in` (only for relations), `isl_dim_out` (only for relations), `isl_dim_set` (only for sets) or `isl_dim_all`.

To check whether a given space is that of a set or a map or whether it is a parameter space, use these functions:

```
#include <isl/space.h>
int isl_space_is_params(__isl_keep isl_space *space);
int isl_space_is_set(__isl_keep isl_space *space);
int isl_space_is_map(__isl_keep isl_space *space);
```

Spaces can be compared using the following functions:

```
#include <isl/space.h>
int isl_space_is_equal(__isl_keep isl_space *space1,
    __isl_keep isl_space *space2);
int isl_space_is_domain(__isl_keep isl_space *space1,
    __isl_keep isl_space *space2);
int isl_space_is_range(__isl_keep isl_space *space1,
    __isl_keep isl_space *space2);
int isl_space_tuple_is_equal(
    __isl_keep isl_space *space1,
    enum isl_dim_type type1,
    __isl_keep isl_space *space2,
    enum isl_dim_type type2);
```

`isl_space_is_domain` checks whether the first argument is equal to the domain of the second argument. This requires in particular that the first argument is a set space and that the second argument is a map space. `isl_space_tuple_is_equal` checks whether the given tuples (`isl_dim_in`, `isl_dim_out` or `isl_dim_set`) of the given spaces are the same. That is, it checks if they have the same identifier (if any), the same dimension and the same internal structure (if any).

It is often useful to create objects that live in the same space as some other object. This can be accomplished by creating the new objects (see §1.4.10 or §1.4.23) based on the space of the original object.

```
#include <isl/set.h>
__isl_give isl_space *isl_basic_set_get_space(
    __isl_keep isl_basic_set *bset);
__isl_give isl_space *isl_set_get_space(__isl_keep isl_set *set);

#include <isl/union_set.h>
__isl_give isl_space *isl_union_set_get_space(
    __isl_keep isl_union_set *uset);
```

```

#include <isl/map.h>
__isl_give isl_space *isl_basic_map_get_space(
    __isl_keep isl_basic_map *bmap);
__isl_give isl_space *isl_map_get_space(__isl_keep isl_map *map);

#include <isl/union_map.h>
__isl_give isl_space *isl_union_map_get_space(
    __isl_keep isl_union_map *umap);

#include <isl/constraint.h>
__isl_give isl_space *isl_constraint_get_space(
    __isl_keep isl_constraint *constraint);

#include <isl/polynomial.h>
__isl_give isl_space *isl_qpolynomial_get_domain_space(
    __isl_keep isl_qpolynomial *qp);
__isl_give isl_space *isl_qpolynomial_get_space(
    __isl_keep isl_qpolynomial *qp);
__isl_give isl_space *isl_qpolynomial_fold_get_space(
    __isl_keep isl_qpolynomial_fold *fold);
__isl_give isl_space *isl_pw_qpolynomial_get_domain_space(
    __isl_keep isl_pw_qpolynomial *pwqp);
__isl_give isl_space *isl_pw_qpolynomial_get_space(
    __isl_keep isl_pw_qpolynomial *pwqp);
__isl_give isl_space *isl_pw_qpolynomial_fold_get_domain_space(
    __isl_keep isl_pw_qpolynomial_fold *pwf);
__isl_give isl_space *isl_pw_qpolynomial_fold_get_space(
    __isl_keep isl_pw_qpolynomial_fold *pwf);
__isl_give isl_space *isl_union_pw_qpolynomial_get_space(
    __isl_keep isl_union_pw_qpolynomial *upwqp);
__isl_give isl_space *isl_union_pw_qpolynomial_fold_get_space(
    __isl_keep isl_union_pw_qpolynomial_fold *upwf);

#include <isl/val.h>
__isl_give isl_space *isl_multi_val_get_space(
    __isl_keep isl_multi_val *mv);

#include <isl/aff.h>
__isl_give isl_space *isl_aff_get_domain_space(
    __isl_keep isl_aff *aff);
__isl_give isl_space *isl_aff_get_space(
    __isl_keep isl_aff *aff);
__isl_give isl_space *isl_pw_aff_get_domain_space(
    __isl_keep isl_pw_aff *pwaff);
__isl_give isl_space *isl_pw_aff_get_space(
    __isl_keep isl_pw_aff *pwaff);
__isl_give isl_space *isl_multi_aff_get_domain_space(

```

```

        __isl_keep isl_multi_aff *maff);
__isl_give isl_space *isl_multi_aff_get_space(
        __isl_keep isl_multi_aff *maff);
__isl_give isl_space *isl_pw_multi_aff_get_domain_space(
        __isl_keep isl_pw_multi_aff *pma);
__isl_give isl_space *isl_pw_multi_aff_get_space(
        __isl_keep isl_pw_multi_aff *pma);
__isl_give isl_space *isl_union_pw_multi_aff_get_space(
        __isl_keep isl_union_pw_multi_aff *upma);
__isl_give isl_space *isl_multi_pw_aff_get_domain_space(
        __isl_keep isl_multi_pw_aff *mpa);
__isl_give isl_space *isl_multi_pw_aff_get_space(
        __isl_keep isl_multi_pw_aff *mpa);

#include <isl/point.h>
__isl_give isl_space *isl_point_get_space(
        __isl_keep isl_point *pnt);

```

The identifiers or names of the individual dimensions may be set or read off using the following functions.

```

#include <isl/space.h>
__isl_give isl_space *isl_space_set_dim_id(
        __isl_take isl_space *space,
        enum isl_dim_type type, unsigned pos,
        __isl_take isl_id *id);
int isl_space_has_dim_id(__isl_keep isl_space *space,
        enum isl_dim_type type, unsigned pos);
__isl_give isl_id *isl_space_get_dim_id(
        __isl_keep isl_space *space,
        enum isl_dim_type type, unsigned pos);
__isl_give isl_space *isl_space_set_dim_name(
        __isl_take isl_space *space,
        enum isl_dim_type type, unsigned pos,
        __isl_keep const char *name);
int isl_space_has_dim_name(__isl_keep isl_space *space,
        enum isl_dim_type type, unsigned pos);
__isl_keep const char *isl_space_get_dim_name(
        __isl_keep isl_space *space,
        enum isl_dim_type type, unsigned pos);

```

Note that `isl_space_get_name` returns a pointer to some internal data structure, so the result can only be used while the corresponding `isl_space` is alive. Also note that every function that operates on two sets or relations requires that both arguments have the same parameters. This also means that if one of the arguments has named parameters, then the other needs to have named parameters too and the names need to match. Pairs of `isl_set`, `isl_map`, `isl_union_set` and/or `isl_union_map` arguments

may have different parameters (as long as they are named), in which case the result will have as parameters the union of the parameters of the arguments.

Given the identifier or name of a dimension (typically a parameter), its position can be obtained from the following function.

```
#include <isl/space.h>
int isl_space_find_dim_by_id(__isl_keep isl_space *space,
    enum isl_dim_type type, __isl_keep isl_id *id);
int isl_space_find_dim_by_name(__isl_keep isl_space *space,
    enum isl_dim_type type, const char *name);
```

The identifiers or names of entire spaces may be set or read off using the following functions.

```
#include <isl/space.h>
__isl_give isl_space *isl_space_set_tuple_id(
    __isl_take isl_space *space,
    enum isl_dim_type type, __isl_take isl_id *id);
__isl_give isl_space *isl_space_reset_tuple_id(
    __isl_take isl_space *space, enum isl_dim_type type);
int isl_space_has_tuple_id(__isl_keep isl_space *space,
    enum isl_dim_type type);
__isl_give isl_id *isl_space_get_tuple_id(
    __isl_keep isl_space *space, enum isl_dim_type type);
__isl_give isl_space *isl_space_set_tuple_name(
    __isl_take isl_space *space,
    enum isl_dim_type type, const char *s);
int isl_space_has_tuple_name(__isl_keep isl_space *space,
    enum isl_dim_type type);
const char *isl_space_get_tuple_name(__isl_keep isl_space *space,
    enum isl_dim_type type);
```

The type argument needs to be one of `isl.dim.in`, `isl.dim.out` or `isl.dim.set`. As with `isl_space_get_name`, the `isl_space_get_tuple_name` function returns a pointer to some internal data structure. Binary operations require the corresponding spaces of their arguments to have the same name.

To keep the names of all parameters and tuples, but reset the user pointers of all the corresponding identifiers, use the following function.

```
__isl_give isl_space *isl_space_reset_user(
    __isl_take isl_space *space);
```

Spaces can be nested. In particular, the domain of a set or the domain or range of a relation can be a nested relation. This process is also called *wrapping*. The functions for detecting, constructing and deconstructing such nested spaces can be found in the wrapping properties of §1.4.12, the wrapping operations of §1.4.13 and the Cartesian product operations of §1.4.14.

Spaces can be created from other spaces using the following functions.

```

__isl_give isl_space *isl_space_domain(__isl_take isl_space *space);
__isl_give isl_space *isl_space_from_domain(__isl_take isl_space *space);
__isl_give isl_space *isl_space_range(__isl_take isl_space *space);
__isl_give isl_space *isl_space_from_range(__isl_take isl_space *space);
__isl_give isl_space *isl_space_domain_map(
    __isl_take isl_space *space);
__isl_give isl_space *isl_space_range_map(
    __isl_take isl_space *space);
__isl_give isl_space *isl_space_params(
    __isl_take isl_space *space);
__isl_give isl_space *isl_space_set_from_params(
    __isl_take isl_space *space);
__isl_give isl_space *isl_space_reverse(__isl_take isl_space *space);
__isl_give isl_space *isl_space_join(__isl_take isl_space *left,
    __isl_take isl_space *right);
__isl_give isl_space *isl_space_align_params(
    __isl_take isl_space *space1, __isl_take isl_space *space2)
__isl_give isl_space *isl_space_insert_dims(__isl_take isl_space *space,
    enum isl_dim_type type, unsigned pos, unsigned n);
__isl_give isl_space *isl_space_add_dims(__isl_take isl_space *space,
    enum isl_dim_type type, unsigned n);
__isl_give isl_space *isl_space_drop_dims(__isl_take isl_space *space,
    enum isl_dim_type type, unsigned first, unsigned n);
__isl_give isl_space *isl_space_move_dims(__isl_take isl_space *space,
    enum isl_dim_type dst_type, unsigned dst_pos,
    enum isl_dim_type src_type, unsigned src_pos,
    unsigned n);
__isl_give isl_space *isl_space_map_from_set(
    __isl_take isl_space *space);
__isl_give isl_space *isl_space_map_from_domain_and_range(
    __isl_take isl_space *domain,
    __isl_take isl_space *range);
__isl_give isl_space *isl_space_zip(__isl_take isl_space *space);
__isl_give isl_space *isl_space_curry(
    __isl_take isl_space *space);
__isl_give isl_space *isl_space_uncurry(
    __isl_take isl_space *space);

```

Note that if dimensions are added or removed from a space, then the name and the internal structure are lost.

1.4.8 Local Spaces

A local space is essentially a space with zero or more existentially quantified variables. The local space of a (constraint of a) basic set or relation can be obtained using the following functions.

```

#include <isl/constraint.h>
__isl_give isl_local_space *isl_constraint_get_local_space(
    __isl_keep isl_constraint *constraint);

#include <isl/set.h>
__isl_give isl_local_space *isl_basic_set_get_local_space(
    __isl_keep isl_basic_set *bset);

#include <isl/map.h>
__isl_give isl_local_space *isl_basic_map_get_local_space(
    __isl_keep isl_basic_map *bmap);

```

A new local space can be created from a space using

```

#include <isl/local_space.h>
__isl_give isl_local_space *isl_local_space_from_space(
    __isl_take isl_space *space);

```

They can be inspected, modified, copied and freed using the following functions.

```

#include <isl/local_space.h>
isl_ctx *isl_local_space_get_ctx(
    __isl_keep isl_local_space *ls);
int isl_local_space_is_set(__isl_keep isl_local_space *ls);
int isl_local_space_dim(__isl_keep isl_local_space *ls,
    enum isl_dim_type type);
__isl_give isl_local_space *isl_local_space_set_tuple_id(
    __isl_take isl_local_space *ls,
    enum isl_dim_type type, __isl_take isl_id *id);
int isl_local_space_has_dim_id(
    __isl_keep isl_local_space *ls,
    enum isl_dim_type type, unsigned pos);
__isl_give isl_id *isl_local_space_get_dim_id(
    __isl_keep isl_local_space *ls,
    enum isl_dim_type type, unsigned pos);
int isl_local_space_has_dim_name(
    __isl_keep isl_local_space *ls,
    enum isl_dim_type type, unsigned pos)
const char *isl_local_space_get_dim_name(
    __isl_keep isl_local_space *ls,
    enum isl_dim_type type, unsigned pos);
__isl_give isl_local_space *isl_local_space_set_dim_name(
    __isl_take isl_local_space *ls,
    enum isl_dim_type type, unsigned pos, const char *s);
__isl_give isl_local_space *isl_local_space_set_dim_id(
    __isl_take isl_local_space *ls,
    enum isl_dim_type type, unsigned pos,

```

```

        __isl_take isl_id *id);
__isl_give isl_space *isl_local_space_get_space(
    __isl_keep isl_local_space *ls);
__isl_give isl_aff *isl_local_space_get_div(
    __isl_keep isl_local_space *ls, int pos);
__isl_give isl_local_space *isl_local_space_copy(
    __isl_keep isl_local_space *ls);
__isl_null isl_local_space *isl_local_space_free(
    __isl_take isl_local_space *ls);

```

Note that `isl_local_space_get_div` can only be used on local spaces of sets.
Two local spaces can be compared using

```

int isl_local_space_is_equal(__isl_keep isl_local_space *ls1,
    __isl_keep isl_local_space *ls2);

```

Local spaces can be created from other local spaces using the functions described in §1.4.13 and §1.4.14.

1.4.9 Input and Output

`isl` supports its own input/output format, which is similar to the *Omega* format, but also supports the *PolyLib* format in some cases.

isl format

The `isl` format is similar to that of *Omega*, but has a different syntax for describing the parameters and allows for the definition of an existentially quantified variable as the integer division of an affine expression. For example, the set of integers i between 0 and n such that $i \% 10 \leq 6$ can be described as

```

[n] -> { [i] : exists (a = [i/10] : 0 <= i and i <= n and
                    i - 10 a <= 6) }

```

A set or relation can have several disjuncts, separated by the keyword `or`. Each disjunct is either a conjunction of constraints or a projection (`exists`) of a conjunction of constraints. The constraints are separated by the keyword `and`.

PolyLib format

If the represented set is a union, then the first line contains a single number representing the number of disjuncts. Otherwise, a line containing the number 1 is optional.

Each disjunct is represented by a matrix of constraints. The first line contains two numbers representing the number of rows and columns, where the number of rows is equal to the number of constraints and the number of columns is equal to two plus the number of variables. The following lines contain the actual rows of the constraint matrix. In each row, the first column indicates whether the constraint is an equality (0) or inequality (1). The final column corresponds to the constant term.

If the set is parametric, then the coefficients of the parameters appear in the last columns before the constant column. The coefficients of any existentially quantified variables appear between those of the set variables and those of the parameters.

Extended PolyLib format

The extended PolyLib format is nearly identical to the PolyLib format. The only difference is that the line containing the number of rows and columns of a constraint matrix also contains four additional numbers: the number of output dimensions, the number of input dimensions, the number of local dimensions (i.e., the number of existentially quantified variables) and the number of parameters. For sets, the number of “output” dimensions is equal to the number of set dimensions, while the number of “input” dimensions is zero.

Input

```
#include <isl/set.h>
__isl_give isl_basic_set *isl_basic_set_read_from_file(
    isl_ctx *ctx, FILE *input);
__isl_give isl_basic_set *isl_basic_set_read_from_str(
    isl_ctx *ctx, const char *str);
__isl_give isl_set *isl_set_read_from_file(isl_ctx *ctx,
    FILE *input);
__isl_give isl_set *isl_set_read_from_str(isl_ctx *ctx,
    const char *str);

#include <isl/map.h>
__isl_give isl_basic_map *isl_basic_map_read_from_file(
    isl_ctx *ctx, FILE *input);
__isl_give isl_basic_map *isl_basic_map_read_from_str(
    isl_ctx *ctx, const char *str);
__isl_give isl_map *isl_map_read_from_file(
    isl_ctx *ctx, FILE *input);
__isl_give isl_map *isl_map_read_from_str(isl_ctx *ctx,
    const char *str);

#include <isl/union_set.h>
__isl_give isl_union_set *isl_union_set_read_from_file(
    isl_ctx *ctx, FILE *input);
__isl_give isl_union_set *isl_union_set_read_from_str(
    isl_ctx *ctx, const char *str);

#include <isl/union_map.h>
__isl_give isl_union_map *isl_union_map_read_from_file(
    isl_ctx *ctx, FILE *input);
__isl_give isl_union_map *isl_union_map_read_from_str(
    isl_ctx *ctx, const char *str);
```


The input format is autodetected and may be either the PolyLib format or the isl format.

Output

Before anything can be printed, an `isl_printer` needs to be created.

```
__isl_give isl_printer *isl_printer_to_file(isl_ctx *ctx,
      FILE *file);
__isl_give isl_printer *isl_printer_to_str(isl_ctx *ctx);
__isl_null isl_printer *isl_printer_free(
      __isl_take isl_printer *printer);
__isl_give char *isl_printer_get_str(
      __isl_keep isl_printer *printer);
```

The printer can be inspected using the following functions.

```
FILE *isl_printer_get_file(
      __isl_keep isl_printer *printer);
int isl_printer_get_output_format(
      __isl_keep isl_printer *p);
```

The behavior of the printer can be modified in various ways

```
__isl_give isl_printer *isl_printer_set_output_format(
      __isl_take isl_printer *p, int output_format);
__isl_give isl_printer *isl_printer_set_indent(
      __isl_take isl_printer *p, int indent);
__isl_give isl_printer *isl_printer_set_indent_prefix(
      __isl_take isl_printer *p, const char *prefix);
__isl_give isl_printer *isl_printer_indent(
      __isl_take isl_printer *p, int indent);
__isl_give isl_printer *isl_printer_set_prefix(
      __isl_take isl_printer *p, const char *prefix);
__isl_give isl_printer *isl_printer_set_suffix(
      __isl_take isl_printer *p, const char *suffix);
```

The `output_format` may be either `ISL_FORMAT_ISL`, `ISL_FORMAT_OMEGA`, `ISL_FORMAT_POLYLIB`, `ISL_FORMAT_EXT_POLYLIB` or `ISL_FORMAT_LATEX` and defaults to `ISL_FORMAT_ISL`. Each line in the output is prefixed by `indent_prefix`, indented by `indent` (set by `isl_printer_set_indent`) spaces (default: 0), prefixed by `prefix` and suffixed by `suffix`. In the PolyLib format output, the coefficients of the existentially quantified variables appear between those of the set variables and those of the parameters. The function `isl_printer_indent` increases the indentation by the specified amount (which may be negative).

To actually print something, use

```

#include <isl/printer.h>
__isl_give isl_printer *isl_printer_print_double(
    __isl_take isl_printer *p, double d);

#include <isl/set.h>
__isl_give isl_printer *isl_printer_print_basic_set(
    __isl_take isl_printer *printer,
    __isl_keep isl_basic_set *bset);
__isl_give isl_printer *isl_printer_print_set(
    __isl_take isl_printer *printer,
    __isl_keep isl_set *set);

#include <isl/map.h>
__isl_give isl_printer *isl_printer_print_basic_map(
    __isl_take isl_printer *printer,
    __isl_keep isl_basic_map *bmap);
__isl_give isl_printer *isl_printer_print_map(
    __isl_take isl_printer *printer,
    __isl_keep isl_map *map);

#include <isl/union_set.h>
__isl_give isl_printer *isl_printer_print_union_set(
    __isl_take isl_printer *p,
    __isl_keep isl_union_set *uset);

#include <isl/union_map.h>
__isl_give isl_printer *isl_printer_print_union_map(
    __isl_take isl_printer *p,
    __isl_keep isl_union_map *umap);

```

When called on a file printer, the following function flushes the file. When called on a string printer, the buffer is cleared.

```

__isl_give isl_printer *isl_printer_flush(
    __isl_take isl_printer *p);

```

1.4.10 Creating New Sets and Relations

isl has functions for creating some standard sets and relations.

- Empty sets and relations

```

__isl_give isl_basic_set *isl_basic_set_empty(
    __isl_take isl_space *space);
__isl_give isl_basic_map *isl_basic_map_empty(
    __isl_take isl_space *space);
__isl_give isl_set *isl_set_empty(

```

```

        __isl_take isl_space *space);
__isl_give isl_map *isl_map_empty(
        __isl_take isl_space *space);
__isl_give isl_union_set *isl_union_set_empty(
        __isl_take isl_space *space);
__isl_give isl_union_map *isl_union_map_empty(
        __isl_take isl_space *space);

```

For `isl_union_sets` and `isl_union_maps`, the space is only used to specify the parameters.

- Universe sets and relations

```

__isl_give isl_basic_set *isl_basic_set_universe(
        __isl_take isl_space *space);
__isl_give isl_basic_map *isl_basic_map_universe(
        __isl_take isl_space *space);
__isl_give isl_set *isl_set_universe(
        __isl_take isl_space *space);
__isl_give isl_map *isl_map_universe(
        __isl_take isl_space *space);
__isl_give isl_union_set *isl_union_set_universe(
        __isl_take isl_union_set *uset);
__isl_give isl_union_map *isl_union_map_universe(
        __isl_take isl_union_map *umap);

```

The sets and relations constructed by the functions above contain all integer values, while those constructed by the functions below only contain non-negative values.

```

__isl_give isl_basic_set *isl_basic_set_nat_universe(
        __isl_take isl_space *space);
__isl_give isl_basic_map *isl_basic_map_nat_universe(
        __isl_take isl_space *space);
__isl_give isl_set *isl_set_nat_universe(
        __isl_take isl_space *space);
__isl_give isl_map *isl_map_nat_universe(
        __isl_take isl_space *space);

```

- Identity relations

```

__isl_give isl_basic_map *isl_basic_map_identity(
        __isl_take isl_space *space);
__isl_give isl_map *isl_map_identity(
        __isl_take isl_space *space);

```

The number of input and output dimensions in `space` needs to be the same.

- Lexicographic order

```

__isl_give isl_map *isl_map_lex_lt(
    __isl_take isl_space *set_space);
__isl_give isl_map *isl_map_lex_le(
    __isl_take isl_space *set_space);
__isl_give isl_map *isl_map_lex_gt(
    __isl_take isl_space *set_space);
__isl_give isl_map *isl_map_lex_ge(
    __isl_take isl_space *set_space);
__isl_give isl_map *isl_map_lex_lt_first(
    __isl_take isl_space *space, unsigned n);
__isl_give isl_map *isl_map_lex_le_first(
    __isl_take isl_space *space, unsigned n);
__isl_give isl_map *isl_map_lex_gt_first(
    __isl_take isl_space *space, unsigned n);
__isl_give isl_map *isl_map_lex_ge_first(
    __isl_take isl_space *space, unsigned n);

```

The first four functions take a space for a **set** and return relations that express that the elements in the domain are lexicographically less (`isl_map_lex_lt`), less or equal (`isl_map_lex_le`), greater (`isl_map_lex_gt`) or greater or equal (`isl_map_lex_ge`) than the elements in the range. The last four functions take a space for a map and return relations that express that the first *n* dimensions in the domain are lexicographically less (`isl_map_lex_lt_first`), less or equal (`isl_map_lex_le_first`), greater (`isl_map_lex_gt_first`) or greater or equal (`isl_map_lex_ge_first`) than the first *n* dimensions in the range.

A basic set or relation can be converted to a set or relation using the following functions.

```

__isl_give isl_set *isl_set_from_basic_set(
    __isl_take isl_basic_set *bset);
__isl_give isl_map *isl_map_from_basic_map(
    __isl_take isl_basic_map *bmap);

```

Sets and relations can be converted to union sets and relations using the following functions.

```

__isl_give isl_union_set *isl_union_set_from_basic_set(
    __isl_take isl_basic_set *bset);
__isl_give isl_union_map *isl_union_map_from_basic_map(
    __isl_take isl_basic_map *bmap);
__isl_give isl_union_set *isl_union_set_from_set(
    __isl_take isl_set *set);
__isl_give isl_union_map *isl_union_map_from_map(
    __isl_take isl_map *map);

```

The inverse conversions below can only be used if the input union set or relation is known to contain elements in exactly one space.

```
__isl_give isl_set *isl_set_from_union_set(
    __isl_take isl_union_set *uset);
__isl_give isl_map *isl_map_from_union_map(
    __isl_take isl_union_map *umap);
```

A zero-dimensional (basic) set can be constructed on a given parameter domain using the following function.

```
__isl_give isl_basic_set *isl_basic_set_from_params(
    __isl_take isl_basic_set *bset);
__isl_give isl_set *isl_set_from_params(
    __isl_take isl_set *set);
```

Sets and relations can be copied and freed again using the following functions.

```
__isl_give isl_basic_set *isl_basic_set_copy(
    __isl_keep isl_basic_set *bset);
__isl_give isl_set *isl_set_copy(__isl_keep isl_set *set);
__isl_give isl_union_set *isl_union_set_copy(
    __isl_keep isl_union_set *uset);
__isl_give isl_basic_map *isl_basic_map_copy(
    __isl_keep isl_basic_map *bmap);
__isl_give isl_map *isl_map_copy(__isl_keep isl_map *map);
__isl_give isl_union_map *isl_union_map_copy(
    __isl_keep isl_union_map *umap);
__isl_null isl_basic_set *isl_basic_set_free(
    __isl_take isl_basic_set *bset);
__isl_null isl_set *isl_set_free(__isl_take isl_set *set);
__isl_null isl_union_set *isl_union_set_free(
    __isl_take isl_union_set *uset);
__isl_null isl_basic_map *isl_basic_map_free(
    __isl_take isl_basic_map *bmap);
__isl_null isl_map *isl_map_free(__isl_take isl_map *map);
__isl_null isl_union_map *isl_union_map_free(
    __isl_take isl_union_map *umap);
```

Other sets and relations can be constructed by starting from a universe set or relation, adding equality and/or inequality constraints and then projecting out the existentially quantified variables, if any. Constraints can be constructed, manipulated and added to (or removed from) (basic) sets and relations using the following functions.

```
#include <isl/constraint.h>
__isl_give isl_constraint *isl_equality_alloc(
    __isl_take isl_local_space *ls);
```

```

__isl_give isl_constraint *isl_inequality_alloc(
    __isl_take isl_local_space *ls);
__isl_give isl_constraint *isl_constraint_set_constant_si(
    __isl_take isl_constraint *constraint, int v);
__isl_give isl_constraint *isl_constraint_set_constant_val(
    __isl_take isl_constraint *constraint,
    __isl_take isl_val *v);
__isl_give isl_constraint *isl_constraint_set_coefficient_si(
    __isl_take isl_constraint *constraint,
    enum isl_dim_type type, int pos, int v);
__isl_give isl_constraint *
isl_constraint_set_coefficient_val(
    __isl_take isl_constraint *constraint,
    enum isl_dim_type type, int pos,
    __isl_take isl_val *v);
__isl_give isl_basic_map *isl_basic_map_add_constraint(
    __isl_take isl_basic_map *bmap,
    __isl_take isl_constraint *constraint);
__isl_give isl_basic_set *isl_basic_set_add_constraint(
    __isl_take isl_basic_set *bset,
    __isl_take isl_constraint *constraint);
__isl_give isl_map *isl_map_add_constraint(
    __isl_take isl_map *map,
    __isl_take isl_constraint *constraint);
__isl_give isl_set *isl_set_add_constraint(
    __isl_take isl_set *set,
    __isl_take isl_constraint *constraint);
__isl_give isl_basic_set *isl_basic_set_drop_constraint(
    __isl_take isl_basic_set *bset,
    __isl_take isl_constraint *constraint);

```

For example, to create a set containing the even integers between 10 and 42, you would use the following code.

```

isl_space *space;
isl_local_space *ls;
isl_constraint *c;
isl_basic_set *bset;

space = isl_space_set_alloc(ctx, 0, 2);
bset = isl_basic_set_universe(isl_space_copy(space));
ls = isl_local_space_from_space(space);

c = isl_equality_alloc(isl_local_space_copy(ls));
c = isl_constraint_set_coefficient_si(c, isl_dim_set, 0, -1);
c = isl_constraint_set_coefficient_si(c, isl_dim_set, 1, 2);
bset = isl_basic_set_add_constraint(bset, c);

```

```

c = isl_inequality_alloc(isl_local_space_copy(ls));
c = isl_constraint_set_constant_si(c, -10);
c = isl_constraint_set_coefficient_si(c, isl_dim_set, 0, 1);
bset = isl_basic_set_add_constraint(bset, c);

c = isl_inequality_alloc(ls);
c = isl_constraint_set_constant_si(c, 42);
c = isl_constraint_set_coefficient_si(c, isl_dim_set, 0, -1);
bset = isl_basic_set_add_constraint(bset, c);

bset = isl_basic_set_project_out(bset, isl_dim_set, 1, 1);

```

Or, alternatively,

```

isl_basic_set *bset;
bset = isl_basic_set_read_from_str(ctx,
    "{[i] : exists (a : i = 2a and i >= 10 and i <= 42)}");

```

A basic set or relation can also be constructed from two matrices describing the equalities and the inequalities.

```

__isl_give isl_basic_set *isl_basic_set_from_constraint_matrices(
    __isl_take isl_space *space,
    __isl_take isl_mat *eq, __isl_take isl_mat *ineq,
    enum isl_dim_type c1,
    enum isl_dim_type c2, enum isl_dim_type c3,
    enum isl_dim_type c4);
__isl_give isl_basic_map *isl_basic_map_from_constraint_matrices(
    __isl_take isl_space *space,
    __isl_take isl_mat *eq, __isl_take isl_mat *ineq,
    enum isl_dim_type c1,
    enum isl_dim_type c2, enum isl_dim_type c3,
    enum isl_dim_type c4, enum isl_dim_type c5);

```

The `isl_dim_type` arguments indicate the order in which different kinds of variables appear in the input matrices and should be a permutation of `isl_dim_cst`, `isl_dim_param`, `isl_dim_set` and `isl_dim_div` for sets and of `isl_dim_cst`, `isl_dim_param`, `isl_dim_in`, `isl_dim_out` and `isl_dim_div` for relations.

A (basic or union) set or relation can also be constructed from a (union) (piecewise) (multiple) affine expression or a list of affine expressions (See §1.4.20 and §1.4.21).

```

__isl_give isl_basic_map *isl_basic_map_from_aff(
    __isl_take isl_aff *aff);
__isl_give isl_map *isl_map_from_aff(
    __isl_take isl_aff *aff);
__isl_give isl_set *isl_set_from_pw_aff(
    __isl_take isl_pw_aff *pwaff);

```

```

__isl_give isl_map *isl_map_from_pw_aff(
    __isl_take isl_pw_aff *pwaff);
__isl_give isl_basic_map *isl_basic_map_from_aff_list(
    __isl_take isl_space *domain_space,
    __isl_take isl_aff_list *list);
__isl_give isl_basic_map *isl_basic_map_from_multi_aff(
    __isl_take isl_multi_aff *maff)
__isl_give isl_map *isl_map_from_multi_aff(
    __isl_take isl_multi_aff *maff)
__isl_give isl_set *isl_set_from_pw_multi_aff(
    __isl_take isl_pw_multi_aff *pma);
__isl_give isl_map *isl_map_from_pw_multi_aff(
    __isl_take isl_pw_multi_aff *pma);
__isl_give isl_set *isl_set_from_multi_pw_aff(
    __isl_take isl_multi_pw_aff *mpa);
__isl_give isl_map *isl_map_from_multi_pw_aff(
    __isl_take isl_multi_pw_aff *mpa);
__isl_give isl_union_map *
isl_union_map_from_union_pw_multi_aff(
    __isl_take isl_union_pw_multi_aff *upma);

```

The `domain_space` argument describes the domain of the resulting basic relation. It is required because the `list` may consist of zero affine expressions.

1.4.11 Inspecting Sets and Relations

Usually, the user should not have to care about the actual constraints of the sets and maps, but should instead apply the abstract operations explained in the following sections. Occasionally, however, it may be required to inspect the individual coefficients of the constraints. This section explains how to do so. In these cases, it may also be useful to have `isl` compute an explicit representation of the existentially quantified variables.

```

__isl_give isl_set *isl_set_compute_divs(
    __isl_take isl_set *set);
__isl_give isl_map *isl_map_compute_divs(
    __isl_take isl_map *map);
__isl_give isl_union_set *isl_union_set_compute_divs(
    __isl_take isl_union_set *uset);
__isl_give isl_union_map *isl_union_map_compute_divs(
    __isl_take isl_union_map *umap);

```

This explicit representation defines the existentially quantified variables as integer divisions of the other variables, possibly including earlier existentially quantified variables. An explicitly represented existentially quantified variable therefore has a unique value when the values of the other variables are known. If, furthermore, the same existentials, i.e., existentials with the same explicit representations, should appear in the

same order in each of the disjuncts of a set or map, then the user should call either of the following functions.

```
__isl_give isl_set *isl_set_align_divs(
    __isl_take isl_set *set);
__isl_give isl_map *isl_map_align_divs(
    __isl_take isl_map *map);
```

Alternatively, the existentially quantified variables can be removed using the following functions, which compute an overapproximation.

```
__isl_give isl_basic_set *isl_basic_set_remove_divs(
    __isl_take isl_basic_set *bset);
__isl_give isl_basic_map *isl_basic_map_remove_divs(
    __isl_take isl_basic_map *bmap);
__isl_give isl_set *isl_set_remove_divs(
    __isl_take isl_set *set);
__isl_give isl_map *isl_map_remove_divs(
    __isl_take isl_map *map);
```

It is also possible to only remove those divs that are defined in terms of a given range of dimensions or only those for which no explicit representation is known.

```
__isl_give isl_basic_set *
isl_basic_set_remove_divs_involving_dims(
    __isl_take isl_basic_set *bset,
    enum isl_dim_type type,
    unsigned first, unsigned n);
__isl_give isl_basic_map *
isl_basic_map_remove_divs_involving_dims(
    __isl_take isl_basic_map *bmap,
    enum isl_dim_type type,
    unsigned first, unsigned n);
__isl_give isl_set *isl_set_remove_divs_involving_dims(
    __isl_take isl_set *set, enum isl_dim_type type,
    unsigned first, unsigned n);
__isl_give isl_map *isl_map_remove_divs_involving_dims(
    __isl_take isl_map *map, enum isl_dim_type type,
    unsigned first, unsigned n);

__isl_give isl_basic_set *
isl_basic_set_remove_unknown_divs(
    __isl_take isl_basic_set *bset);
__isl_give isl_set *isl_set_remove_unknown_divs(
    __isl_take isl_set *set);
__isl_give isl_map *isl_map_remove_unknown_divs(
    __isl_take isl_map *map);
```

To iterate over all the sets or maps in a union set or map, use

```
int isl_union_set_foreach_set(__isl_keep isl_union_set *uset,
    int (*fn)(__isl_take isl_set *set, void *user),
    void *user);
int isl_union_map_foreach_map(__isl_keep isl_union_map *umap,
    int (*fn)(__isl_take isl_map *map, void *user),
    void *user);
```

The number of sets or maps in a union set or map can be obtained from

```
int isl_union_set_n_set(__isl_keep isl_union_set *uset);
int isl_union_map_n_map(__isl_keep isl_union_map *umap);
```

To extract the set or map in a given space from a union, use

```
__isl_give isl_set *isl_union_set_extract_set(
    __isl_keep isl_union_set *uset,
    __isl_take isl_space *space);
__isl_give isl_map *isl_union_map_extract_map(
    __isl_keep isl_union_map *umap,
    __isl_take isl_space *space);
```

To iterate over all the basic sets or maps in a set or map, use

```
int isl_set_foreach_basic_set(__isl_keep isl_set *set,
    int (*fn)(__isl_take isl_basic_set *bset, void *user),
    void *user);
int isl_map_foreach_basic_map(__isl_keep isl_map *map,
    int (*fn)(__isl_take isl_basic_map *bmap, void *user),
    void *user);
```

The callback function `fn` should return 0 if successful and -1 if an error occurs. In the latter case, or if any other error occurs, the above functions will return -1.

It should be noted that `isl` does not guarantee that the basic sets or maps passed to `fn` are disjoint. If this is required, then the user should call one of the following functions first.

```
__isl_give isl_set *isl_set_make_disjoint(
    __isl_take isl_set *set);
__isl_give isl_map *isl_map_make_disjoint(
    __isl_take isl_map *map);
```

The number of basic sets in a set can be obtained or the number of basic maps in a map can be obtained from

```
#include <isl/set.h>
int isl_set_n_basic_set(__isl_keep isl_set *set);
```

```
#include <isl/map.h>
int isl_map_n_basic_map(__isl_keep isl_map *map);
```

To iterate over the constraints of a basic set or map, use

```
#include <isl/constraint.h>

int isl_basic_set_n_constraint(
    __isl_keep isl_basic_set *bset);
int isl_basic_set_foreach_constraint(
    __isl_keep isl_basic_set *bset,
    int (*fn)(__isl_take isl_constraint *c, void *user),
    void *user);
int isl_basic_map_n_constraint(
    __isl_keep isl_basic_map *bmap);
int isl_basic_map_foreach_constraint(
    __isl_keep isl_basic_map *bmap,
    int (*fn)(__isl_take isl_constraint *c, void *user),
    void *user);
__isl_null isl_constraint *isl_constraint_free(
    __isl_take isl_constraint *c);
```

Again, the callback function `fn` should return 0 if successful and -1 if an error occurs. In the latter case, or if any other error occurs, the above functions will return -1. The constraint `c` represents either an equality or an inequality. Use the following function to find out whether a constraint represents an equality. If not, it represents an inequality.

```
int isl_constraint_is_equality(
    __isl_keep isl_constraint *constraint);
```

It is also possible to obtain a list of constraints from a basic map or set

```
#include <isl/constraint.h>
__isl_give isl_constraint_list *
isl_basic_map_get_constraint_list(
    __isl_keep isl_basic_map *bmap);
__isl_give isl_constraint_list *
isl_basic_set_get_constraint_list(
    __isl_keep isl_basic_set *bset);
```

These functions require that all existentially quantified variables have an explicit representation. The returned list can be manipulated using the functions in §1.4.15.

The coefficients of the constraints can be inspected using the following functions.

```
int isl_constraint_is_lower_bound(
    __isl_keep isl_constraint *constraint,
    enum isl_dim_type type, unsigned pos);
```

```

int isl_constraint_is_upper_bound(
    __isl_keep isl_constraint *constraint,
    enum isl_dim_type type, unsigned pos);
__isl_give isl_val *isl_constraint_get_constant_val(
    __isl_keep isl_constraint *constraint);
__isl_give isl_val *isl_constraint_get_coefficient_val(
    __isl_keep isl_constraint *constraint,
    enum isl_dim_type type, int pos);
int isl_constraint_involves_dims(
    __isl_keep isl_constraint *constraint,
    enum isl_dim_type type, unsigned first, unsigned n);

```

The explicit representations of the existentially quantified variables can be inspected using the following function. Note that the user is only allowed to use this function if the inspected set or map is the result of a call to `isl_set_compute_divs` or `isl_map_compute_divs`. The existentially quantified variable is equal to the floor of the returned affine expression. The affine expression itself can be inspected using the functions in §1.4.20.

```

__isl_give isl_aff *isl_constraint_get_div(
    __isl_keep isl_constraint *constraint, int pos);

```

To obtain the constraints of a basic set or map in matrix form, use the following functions.

```

__isl_give isl_mat *isl_basic_set_equalities_matrix(
    __isl_keep isl_basic_set *bset,
    enum isl_dim_type c1, enum isl_dim_type c2,
    enum isl_dim_type c3, enum isl_dim_type c4);
__isl_give isl_mat *isl_basic_set_inequalities_matrix(
    __isl_keep isl_basic_set *bset,
    enum isl_dim_type c1, enum isl_dim_type c2,
    enum isl_dim_type c3, enum isl_dim_type c4);
__isl_give isl_mat *isl_basic_map_equalities_matrix(
    __isl_keep isl_basic_map *bmap,
    enum isl_dim_type c1,
    enum isl_dim_type c2, enum isl_dim_type c3,
    enum isl_dim_type c4, enum isl_dim_type c5);
__isl_give isl_mat *isl_basic_map_inequalities_matrix(
    __isl_keep isl_basic_map *bmap,
    enum isl_dim_type c1,
    enum isl_dim_type c2, enum isl_dim_type c3,
    enum isl_dim_type c4, enum isl_dim_type c5);

```

The `isl_dim_type` arguments dictate the order in which different kinds of variables appear in the resulting matrix. For set inputs, they should be a permutation of `isl_dim_cst`, `isl_dim_param`, `isl_dim_set` and `isl_dim_div`. For map inputs, they

should be a permutation of `isl_dim_cst`, `isl_dim_param`, `isl_dim_in`, `isl_dim_out` and `isl_dim_div`.

The number of parameters, input, output or set dimensions can be obtained using the following functions.

```
unsigned isl_basic_set_dim(__isl_keep isl_basic_set *bset,
    enum isl_dim_type type);
unsigned isl_basic_map_dim(__isl_keep isl_basic_map *bmap,
    enum isl_dim_type type);
unsigned isl_set_dim(__isl_keep isl_set *set,
    enum isl_dim_type type);
unsigned isl_map_dim(__isl_keep isl_map *map,
    enum isl_dim_type type);
unsigned isl_union_map_dim(__isl_keep isl_union_map *umap,
    enum isl_dim_type type);
```

Note that a `isl_union_map` only has parameters.

To check whether the description of a set or relation depends on one or more given dimensions, it is not necessary to iterate over all constraints. Instead the following functions can be used.

```
int isl_basic_set_involves_dims(
    __isl_keep isl_basic_set *bset,
    enum isl_dim_type type, unsigned first, unsigned n);
int isl_set_involves_dims(__isl_keep isl_set *set,
    enum isl_dim_type type, unsigned first, unsigned n);
int isl_basic_map_involves_dims(
    __isl_keep isl_basic_map *bmap,
    enum isl_dim_type type, unsigned first, unsigned n);
int isl_map_involves_dims(__isl_keep isl_map *map,
    enum isl_dim_type type, unsigned first, unsigned n);
```

Similarly, the following functions can be used to check whether a given dimension is involved in any lower or upper bound.

```
int isl_set_dim_has_any_lower_bound(__isl_keep isl_set *set,
    enum isl_dim_type type, unsigned pos);
int isl_set_dim_has_any_upper_bound(__isl_keep isl_set *set,
    enum isl_dim_type type, unsigned pos);
```

Note that these functions return true even if there is a bound on the dimension on only some of the basic sets of set. To check if they have a bound for all of the basic sets in set, use the following functions instead.

```
int isl_set_dim_has_lower_bound(__isl_keep isl_set *set,
    enum isl_dim_type type, unsigned pos);
int isl_set_dim_has_upper_bound(__isl_keep isl_set *set,
    enum isl_dim_type type, unsigned pos);
```

The identifiers or names of the domain and range spaces of a set or relation can be read off or set using the following functions.

```

__isl_give isl_basic_set *isl_basic_set_set_tuple_id(
    __isl_take isl_basic_set *bset,
    __isl_take isl_id *id);
__isl_give isl_set *isl_set_set_tuple_id(
    __isl_take isl_set *set, __isl_take isl_id *id);
__isl_give isl_set *isl_set_reset_tuple_id(
    __isl_take isl_set *set);
int isl_set_has_tuple_id(__isl_keep isl_set *set);
__isl_give isl_id *isl_set_get_tuple_id(
    __isl_keep isl_set *set);
__isl_give isl_basic_map *isl_basic_map_set_tuple_id(
    __isl_take isl_basic_map *bmap,
    enum isl_dim_type type, __isl_take isl_id *id);
__isl_give isl_map *isl_map_set_tuple_id(
    __isl_take isl_map *map, enum isl_dim_type type,
    __isl_take isl_id *id);
__isl_give isl_map *isl_map_reset_tuple_id(
    __isl_take isl_map *map, enum isl_dim_type type);
int isl_map_has_tuple_id(__isl_keep isl_map *map,
    enum isl_dim_type type);
__isl_give isl_id *isl_map_get_tuple_id(
    __isl_keep isl_map *map, enum isl_dim_type type);

const char *isl_basic_set_get_tuple_name(
    __isl_keep isl_basic_set *bset);
__isl_give isl_basic_set *isl_basic_set_set_tuple_name(
    __isl_take isl_basic_set *set, const char *s);
int isl_set_has_tuple_name(__isl_keep isl_set *set);
const char *isl_set_get_tuple_name(
    __isl_keep isl_set *set);
__isl_give isl_set *isl_set_set_tuple_name(
    __isl_take isl_set *set, const char *s);
const char *isl_basic_map_get_tuple_name(
    __isl_keep isl_basic_map *bmap,
    enum isl_dim_type type);
__isl_give isl_basic_map *isl_basic_map_set_tuple_name(
    __isl_take isl_basic_map *bmap,
    enum isl_dim_type type, const char *s);
int isl_map_has_tuple_name(__isl_keep isl_map *map,
    enum isl_dim_type type);
const char *isl_map_get_tuple_name(
    __isl_keep isl_map *map,
    enum isl_dim_type type);

```

```

__isl_give isl_map *isl_map_set_tuple_name(
    __isl_take isl_map *map,
    enum isl_dim_type type, const char *s);

```

As with `isl_space_get_tuple_name`, the value returned points to an internal data structure. The identifiers, positions or names of individual dimensions can be read off using the following functions.

```

__isl_give isl_id *isl_basic_set_get_dim_id(
    __isl_keep isl_basic_set *bset,
    enum isl_dim_type type, unsigned pos);
__isl_give isl_set *isl_set_set_dim_id(
    __isl_take isl_set *set, enum isl_dim_type type,
    unsigned pos, __isl_take isl_id *id);
int isl_set_has_dim_id(__isl_keep isl_set *set,
    enum isl_dim_type type, unsigned pos);
__isl_give isl_id *isl_set_get_dim_id(
    __isl_keep isl_set *set, enum isl_dim_type type,
    unsigned pos);
int isl_basic_map_has_dim_id(
    __isl_keep isl_basic_map *bmap,
    enum isl_dim_type type, unsigned pos);
__isl_give isl_map *isl_map_set_dim_id(
    __isl_take isl_map *map, enum isl_dim_type type,
    unsigned pos, __isl_take isl_id *id);
int isl_map_has_dim_id(__isl_keep isl_map *map,
    enum isl_dim_type type, unsigned pos);
__isl_give isl_id *isl_map_get_dim_id(
    __isl_keep isl_map *map, enum isl_dim_type type,
    unsigned pos);
__isl_give isl_id *isl_union_map_get_dim_id(
    __isl_keep isl_union_map *umap,
    enum isl_dim_type type, unsigned pos);

int isl_set_find_dim_by_id(__isl_keep isl_set *set,
    enum isl_dim_type type, __isl_keep isl_id *id);
int isl_map_find_dim_by_id(__isl_keep isl_map *map,
    enum isl_dim_type type, __isl_keep isl_id *id);
int isl_set_find_dim_by_name(__isl_keep isl_set *set,
    enum isl_dim_type type, const char *name);
int isl_map_find_dim_by_name(__isl_keep isl_map *map,
    enum isl_dim_type type, const char *name);

const char *isl_constraint_get_dim_name(
    __isl_keep isl_constraint *constraint,
    enum isl_dim_type type, unsigned pos);
const char *isl_basic_set_get_dim_name(

```

```

        __isl_keep isl_basic_set *bset,
        enum isl_dim_type type, unsigned pos);
int isl_set_has_dim_name(__isl_keep isl_set *set,
        enum isl_dim_type type, unsigned pos);
const char *isl_set_get_dim_name(
        __isl_keep isl_set *set,
        enum isl_dim_type type, unsigned pos);
const char *isl_basic_map_get_dim_name(
        __isl_keep isl_basic_map *bmap,
        enum isl_dim_type type, unsigned pos);
int isl_map_has_dim_name(__isl_keep isl_map *map,
        enum isl_dim_type type, unsigned pos);
const char *isl_map_get_dim_name(
        __isl_keep isl_map *map,
        enum isl_dim_type type, unsigned pos);

```

These functions are mostly useful to obtain the identifiers, positions or names of the parameters. Identifiers of individual dimensions are essentially only useful for printing. They are ignored by all other operations and may not be preserved across those operations.

The user pointers on all parameters and tuples can be reset using the following functions.

```

#include <isl/set.h>
__isl_give isl_set *isl_set_reset_user(
        __isl_take isl_set *set);
#include <isl/map.h>
__isl_give isl_map *isl_map_reset_user(
        __isl_take isl_map *map);
#include <isl/union_set.h>
__isl_give isl_union_set *isl_union_set_reset_user(
        __isl_take isl_union_set *uset);
#include <isl/union_map.h>
__isl_give isl_union_map *isl_union_map_reset_user(
        __isl_take isl_union_map *umap);

```

1.4.12 Properties

Unary Properties

- Emptiness

The following functions test whether the given set or relation contains any integer points. The “plain” variants do not perform any computations, but simply check if the given set or relation is already known to be empty.

```

int isl_basic_set_plain_is_empty(__isl_keep isl_basic_set *bset);

```



```

int isl_basic_set_is_empty(__isl_keep isl_basic_set *bset);
int isl_set_plain_is_empty(__isl_keep isl_set *set);
int isl_set_is_empty(__isl_keep isl_set *set);
int isl_union_set_is_empty(__isl_keep isl_union_set *uset);
int isl_basic_map_plain_is_empty(__isl_keep isl_basic_map *bmap);
int isl_basic_map_is_empty(__isl_keep isl_basic_map *bmap);
int isl_map_plain_is_empty(__isl_keep isl_map *map);
int isl_map_is_empty(__isl_keep isl_map *map);
int isl_union_map_is_empty(__isl_keep isl_union_map *umap);

```

- Universality

```

int isl_basic_set_is_universe(__isl_keep isl_basic_set *bset);
int isl_basic_map_is_universe(__isl_keep isl_basic_map *bmap);
int isl_set_plain_is_universe(__isl_keep isl_set *set);

```

- Single-valuedness

```

#include <isl/set.h>
int isl_set_is_singleton(__isl_keep isl_set *set);

#include <isl/map.h>
int isl_basic_map_is_single_valued(
    __isl_keep isl_basic_map *bmap);
int isl_map_plain_is_single_valued(
    __isl_keep isl_map *map);
int isl_map_is_single_valued(__isl_keep isl_map *map);

#include <isl/union_map.h>
int isl_union_map_is_single_valued(__isl_keep isl_union_map *umap);

```

- Injectivity

```

int isl_map_plain_is_injective(__isl_keep isl_map *map);
int isl_map_is_injective(__isl_keep isl_map *map);
int isl_union_map_plain_is_injective(
    __isl_keep isl_union_map *umap);
int isl_union_map_is_injective(
    __isl_keep isl_union_map *umap);

```

- Bijectivity

```

int isl_map_is_bijective(__isl_keep isl_map *map);
int isl_union_map_is_bijective(__isl_keep isl_union_map *umap);

```

- Position

```

__isl_give isl_val *
isl_basic_map_plain_get_val_if_fixed(
    __isl_keep isl_basic_map *bmap,
    enum isl_dim_type type, unsigned pos);
__isl_give isl_val *isl_set_plain_get_val_if_fixed(
    __isl_keep isl_set *set,
    enum isl_dim_type type, unsigned pos);
__isl_give isl_val *isl_map_plain_get_val_if_fixed(
    __isl_keep isl_map *map,
    enum isl_dim_type type, unsigned pos);

```

If the set or relation obviously lies on a hyperplane where the given dimension has a fixed value, then return that value. Otherwise return NaN.

- Stride

```

int isl_set_dim_residue_class_val(
    __isl_keep isl_set *set,
    int pos, __isl_give isl_val **modulo,
    __isl_give isl_val **residue);

```

Check if the values of the given set dimension are equal to a fixed value modulo some integer value. If so, assign the modulo to **modulo* and the fixed value to **residue*. If the given dimension attains only a single value, then assign 0 to **modulo* and the fixed value to **residue*. If the dimension does not attain only a single value and if no modulo can be found then assign 1 to **modulo* and 1 to **residue*.

- Space

To check whether a set is a parameter domain, use this function:

```

int isl_set_is_params(__isl_keep isl_set *set);
int isl_union_set_is_params(
    __isl_keep isl_union_set *uset);

```

- Wrapping

The following functions check whether the space of the given (basic) set or relation range is a wrapped relation.

```

#include <isl/space.h>
int isl_space_is_wrapping(
    __isl_keep isl_space *space);
int isl_space_domain_is_wrapping(
    __isl_keep isl_space *space);
int isl_space_range_is_wrapping(
    __isl_keep isl_space *space);

```

```

#include <isl/set.h>
int isl_basic_set_is_wrapping(
    __isl_keep isl_basic_set *bset);
int isl_set_is_wrapping(__isl_keep isl_set *set);

#include <isl/map.h>
int isl_map_domain_is_wrapping(
    __isl_keep isl_map *map);
int isl_map_range_is_wrapping(
    __isl_keep isl_map *map);

```

The input to `isl_space_is_wrapping` should be the space of a set, while that of `isl_space_domain_is_wrapping` and `isl_space_range_is_wrapping` should be the space of a relation.

- Internal Product

```

int isl_basic_map_can_zip(
    __isl_keep isl_basic_map *bmap);
int isl_map_can_zip(__isl_keep isl_map *map);

```

Check whether the product of domain and range of the given relation can be computed, i.e., whether both domain and range are nested relations.

- Currying

```

int isl_basic_map_can_curry(
    __isl_keep isl_basic_map *bmap);
int isl_map_can_curry(__isl_keep isl_map *map);

```

Check whether the domain of the (basic) relation is a wrapped relation.

```

int isl_basic_map_can_uncurry(
    __isl_keep isl_basic_map *bmap);
int isl_map_can_uncurry(__isl_keep isl_map *map);

```

Check whether the range of the (basic) relation is a wrapped relation.

Binary Properties

- Equality

```

int isl_basic_set_plain_is_equal(
    __isl_keep isl_basic_set *bset1,
    __isl_keep isl_basic_set *bset2);
int isl_set_plain_is_equal(__isl_keep isl_set *set1,

```

```

        __isl_keep isl_set *set2);
int isl_set_is_equal(__isl_keep isl_set *set1,
        __isl_keep isl_set *set2);
int isl_union_set_is_equal(
        __isl_keep isl_union_set *uset1,
        __isl_keep isl_union_set *uset2);
int isl_basic_map_is_equal(
        __isl_keep isl_basic_map *bmap1,
        __isl_keep isl_basic_map *bmap2);
int isl_map_is_equal(__isl_keep isl_map *map1,
        __isl_keep isl_map *map2);
int isl_map_plain_is_equal(__isl_keep isl_map *map1,
        __isl_keep isl_map *map2);
int isl_union_map_is_equal(
        __isl_keep isl_union_map *umap1,
        __isl_keep isl_union_map *umap2);

```

- Disjointness

```

int isl_basic_set_is_disjoint(
        __isl_keep isl_basic_set *bset1,
        __isl_keep isl_basic_set *bset2);
int isl_set_plain_is_disjoint(__isl_keep isl_set *set1,
        __isl_keep isl_set *set2);
int isl_set_is_disjoint(__isl_keep isl_set *set1,
        __isl_keep isl_set *set2);
int isl_basic_map_is_disjoint(
        __isl_keep isl_basic_map *bmap1,
        __isl_keep isl_basic_map *bmap2);
int isl_map_is_disjoint(__isl_keep isl_map *map1,
        __isl_keep isl_map *map2);

```

- Subset

```

int isl_basic_set_is_subset(
        __isl_keep isl_basic_set *bset1,
        __isl_keep isl_basic_set *bset2);
int isl_set_is_subset(__isl_keep isl_set *set1,
        __isl_keep isl_set *set2);
int isl_set_is_strict_subset(
        __isl_keep isl_set *set1,
        __isl_keep isl_set *set2);
int isl_union_set_is_subset(
        __isl_keep isl_union_set *uset1,
        __isl_keep isl_union_set *uset2);
int isl_union_set_is_strict_subset(

```

```

        __isl_keep isl_union_set *uset1,
        __isl_keep isl_union_set *uset2);
int isl_basic_map_is_subset(
    __isl_keep isl_basic_map *bmap1,
    __isl_keep isl_basic_map *bmap2);
int isl_basic_map_is_strict_subset(
    __isl_keep isl_basic_map *bmap1,
    __isl_keep isl_basic_map *bmap2);
int isl_map_is_subset(
    __isl_keep isl_map *map1,
    __isl_keep isl_map *map2);
int isl_map_is_strict_subset(
    __isl_keep isl_map *map1,
    __isl_keep isl_map *map2);
int isl_union_map_is_subset(
    __isl_keep isl_union_map *umap1,
    __isl_keep isl_union_map *umap2);
int isl_union_map_is_strict_subset(
    __isl_keep isl_union_map *umap1,
    __isl_keep isl_union_map *umap2);

```

Check whether the first argument is a (strict) subset of the second argument.

- Order

Every comparison function returns a negative value if the first argument is considered smaller than the second, a positive value if the first argument is considered greater and zero if the two constraints are considered the same by the comparison criterion.

```

#include <isl/constraint.h>
int isl_constraint_plain_cmp(
    __isl_keep isl_constraint *c1,
    __isl_keep isl_constraint *c2);

```

This function is useful for sorting `isl_constraint`s. The order depends on the internal representation of the inputs. The order is fixed over different calls to the function (assuming the internal representation of the inputs has not changed), but may change over different versions of `isl`.

```

#include <isl/constraint.h>
int isl_constraint_cmp_last_non_zero(
    __isl_keep isl_constraint *c1,
    __isl_keep isl_constraint *c2);

```

This function can be used to sort constraints that live in the same local space. Constraints that involve “earlier” dimensions or that have a smaller coefficient

for the shared latest dimension are considered smaller than other constraints. This function only defines a **partial** order.

```
#include <isl/set.h>
int isl_set_plain_cmp(__isl_keep isl_set *set1,
    __isl_keep isl_set *set2);
```

This function is useful for sorting `isl_sets`. The order depends on the internal representation of the inputs. The order is fixed over different calls to the function (assuming the internal representation of the inputs has not changed), but may change over different versions of `isl`.

1.4.13 Unary Operations

- Complement

```
__isl_give isl_set *isl_set_complement(
    __isl_take isl_set *set);
__isl_give isl_map *isl_map_complement(
    __isl_take isl_map *map);
```

- Inverse map

```
__isl_give isl_basic_map *isl_basic_map_reverse(
    __isl_take isl_basic_map *bmap);
__isl_give isl_map *isl_map_reverse(
    __isl_take isl_map *map);
__isl_give isl_union_map *isl_union_map_reverse(
    __isl_take isl_union_map *umap);
```

- Projection

```
#include <isl/local_space.h>
__isl_give isl_local_space *isl_local_space_domain(
    __isl_take isl_local_space *ls);
__isl_give isl_local_space *isl_local_space_range(
    __isl_take isl_local_space *ls);

#include <isl/set.h>
__isl_give isl_basic_set *isl_basic_set_project_out(
    __isl_take isl_basic_set *bset,
    enum isl_dim_type type, unsigned first, unsigned n);
__isl_give isl_set *isl_set_project_out(__isl_take isl_set *set,
    enum isl_dim_type type, unsigned first, unsigned n);
__isl_give isl_basic_set *isl_basic_set_params(
    __isl_take isl_basic_set *bset);
__isl_give isl_set *isl_set_params(__isl_take isl_set *set);
```

```

#include <isl/map.h>
__isl_give isl_basic_map *isl_basic_map_project_out(
    __isl_take isl_basic_map *bmap,
    enum isl_dim_type type, unsigned first, unsigned n);
__isl_give isl_map *isl_map_project_out(__isl_take isl_map *map,
    enum isl_dim_type type, unsigned first, unsigned n);
__isl_give isl_basic_set *isl_basic_map_domain(
    __isl_take isl_basic_map *bmap);
__isl_give isl_basic_set *isl_basic_map_range(
    __isl_take isl_basic_map *bmap);
__isl_give isl_set *isl_map_params(__isl_take isl_map *map);
__isl_give isl_set *isl_map_domain(
    __isl_take isl_map *bmap);
__isl_give isl_set *isl_map_range(
    __isl_take isl_map *map);

#include <isl/union_set.h>
__isl_give isl_set *isl_union_set_params(
    __isl_take isl_union_set *uset);

#include <isl/union_map.h>
__isl_give isl_union_map *isl_union_map_project_out(
    __isl_take isl_union_map *umap,
    enum isl_dim_type type, unsigned first, unsigned n);
__isl_give isl_set *isl_union_map_params(
    __isl_take isl_union_map *umap);
__isl_give isl_union_set *isl_union_map_domain(
    __isl_take isl_union_map *umap);
__isl_give isl_union_set *isl_union_map_range(
    __isl_take isl_union_map *umap);

```

The function `isl_union_map_project_out` can only project out parameters.

```

#include <isl/map.h>
__isl_give isl_basic_map *isl_basic_map_domain_map(
    __isl_take isl_basic_map *bmap);
__isl_give isl_basic_map *isl_basic_map_range_map(
    __isl_take isl_basic_map *bmap);
__isl_give isl_map *isl_map_domain_map(__isl_take isl_map *map);
__isl_give isl_map *isl_map_range_map(__isl_take isl_map *map);

#include <isl/union_map.h>
__isl_give isl_union_map *isl_union_map_domain_map(
    __isl_take isl_union_map *umap);
__isl_give isl_union_map *isl_union_map_range_map(
    __isl_take isl_union_map *umap);

```

The functions above construct a (basic, regular or union) relation that maps (a wrapped version of) the input relation to its domain or range.

- Elimination

```
__isl_give isl_basic_set *isl_basic_set_eliminate(
    __isl_take isl_basic_set *bset,
    enum isl_dim_type type,
    unsigned first, unsigned n);
__isl_give isl_set *isl_set_eliminate(
    __isl_take isl_set *set, enum isl_dim_type type,
    unsigned first, unsigned n);
__isl_give isl_basic_map *isl_basic_map_eliminate(
    __isl_take isl_basic_map *bmap,
    enum isl_dim_type type,
    unsigned first, unsigned n);
__isl_give isl_map *isl_map_eliminate(
    __isl_take isl_map *map, enum isl_dim_type type,
    unsigned first, unsigned n);
```

Eliminate the coefficients for the given dimensions from the constraints, without removing the dimensions.

- Constructing a relation from a set

```
#include <isl/local_space.h>
__isl_give isl_local_space *isl_local_space_from_domain(
    __isl_take isl_local_space *ls);

#include <isl/map.h>
__isl_give isl_map *isl_map_from_domain(
    __isl_take isl_set *set);
__isl_give isl_map *isl_map_from_range(
    __isl_take isl_set *set);
```

Create a relation with the given set as domain or range. The range or domain of the created relation is a zero-dimensional flat anonymous space.

- Slicing

```
__isl_give isl_basic_set *isl_basic_set_fix_si(
    __isl_take isl_basic_set *bset,
    enum isl_dim_type type, unsigned pos, int value);
__isl_give isl_basic_set *isl_basic_set_fix_val(
    __isl_take isl_basic_set *bset,
    enum isl_dim_type type, unsigned pos,
    __isl_take isl_val *v);
```



```

__isl_give isl_set *isl_set_fix_si(__isl_take isl_set *set,
    enum isl_dim_type type, unsigned pos, int value);
__isl_give isl_set *isl_set_fix_val(
    __isl_take isl_set *set,
    enum isl_dim_type type, unsigned pos,
    __isl_take isl_val *v);
__isl_give isl_basic_map *isl_basic_map_fix_si(
    __isl_take isl_basic_map *bmap,
    enum isl_dim_type type, unsigned pos, int value);
__isl_give isl_basic_map *isl_basic_map_fix_val(
    __isl_take isl_basic_map *bmap,
    enum isl_dim_type type, unsigned pos,
    __isl_take isl_val *v);
__isl_give isl_map *isl_map_fix_si(__isl_take isl_map *map,
    enum isl_dim_type type, unsigned pos, int value);
__isl_give isl_map *isl_map_fix_val(
    __isl_take isl_map *map,
    enum isl_dim_type type, unsigned pos,
    __isl_take isl_val *v);

```

Intersect the set or relation with the hyperplane where the given dimension has the fixed given value.

```

__isl_give isl_basic_map *isl_basic_map_lower_bound_si(
    __isl_take isl_basic_map *bmap,
    enum isl_dim_type type, unsigned pos, int value);
__isl_give isl_basic_map *isl_basic_map_upper_bound_si(
    __isl_take isl_basic_map *bmap,
    enum isl_dim_type type, unsigned pos, int value);
__isl_give isl_set *isl_set_lower_bound_si(
    __isl_take isl_set *set,
    enum isl_dim_type type, unsigned pos, int value);
__isl_give isl_set *isl_set_lower_bound_val(
    __isl_take isl_set *set,
    enum isl_dim_type type, unsigned pos,
    __isl_take isl_val *value);
__isl_give isl_map *isl_map_lower_bound_si(
    __isl_take isl_map *map,
    enum isl_dim_type type, unsigned pos, int value);
__isl_give isl_set *isl_set_upper_bound_si(
    __isl_take isl_set *set,
    enum isl_dim_type type, unsigned pos, int value);
__isl_give isl_set *isl_set_upper_bound_val(
    __isl_take isl_set *set,
    enum isl_dim_type type, unsigned pos,
    __isl_take isl_val *value);

```

```

__isl_give isl_map *isl_map_upper_bound_si(
    __isl_take isl_map *map,
    enum isl_dim_type type, unsigned pos, int value);

```

Intersect the set or relation with the half-space where the given dimension has a value bounded by the fixed given integer value.

```

__isl_give isl_set *isl_set_equate(__isl_take isl_set *set,
    enum isl_dim_type type1, int pos1,
    enum isl_dim_type type2, int pos2);
__isl_give isl_basic_map *isl_basic_map_equate(
    __isl_take isl_basic_map *bmap,
    enum isl_dim_type type1, int pos1,
    enum isl_dim_type type2, int pos2);
__isl_give isl_map *isl_map_equate(__isl_take isl_map *map,
    enum isl_dim_type type1, int pos1,
    enum isl_dim_type type2, int pos2);

```

Intersect the set or relation with the hyperplane where the given dimensions are equal to each other.

```

__isl_give isl_map *isl_map_oppose(__isl_take isl_map *map,
    enum isl_dim_type type1, int pos1,
    enum isl_dim_type type2, int pos2);

```

Intersect the relation with the hyperplane where the given dimensions have opposite values.

```

__isl_give isl_map *isl_map_order_le(
    __isl_take isl_map *map,
    enum isl_dim_type type1, int pos1,
    enum isl_dim_type type2, int pos2);
__isl_give isl_basic_map *isl_basic_map_order_ge(
    __isl_take isl_basic_map *bmap,
    enum isl_dim_type type1, int pos1,
    enum isl_dim_type type2, int pos2);
__isl_give isl_map *isl_map_order_ge(
    __isl_take isl_map *map,
    enum isl_dim_type type1, int pos1,
    enum isl_dim_type type2, int pos2);
__isl_give isl_map *isl_map_order_lt(__isl_take isl_map *map,
    enum isl_dim_type type1, int pos1,
    enum isl_dim_type type2, int pos2);
__isl_give isl_basic_map *isl_basic_map_order_gt(
    __isl_take isl_basic_map *bmap,

```

```

        enum isl_dim_type type1, int pos1,
        enum isl_dim_type type2, int pos2);
__isl_give isl_map *isl_map_order_gt(__isl_take isl_map *map,
        enum isl_dim_type type1, int pos1,
        enum isl_dim_type type2, int pos2);

```

Intersect the relation with the half-space where the given dimensions satisfy the given ordering.

- Identity

```

__isl_give isl_map *isl_set_identity(
    __isl_take isl_set *set);
__isl_give isl_union_map *isl_union_set_identity(
    __isl_take isl_union_set *uset);

```

Construct an identity relation on the given (union) set.

- Deltas

```

__isl_give isl_basic_set *isl_basic_map_deltas(
    __isl_take isl_basic_map *bmap);
__isl_give isl_set *isl_map_deltas(__isl_take isl_map *map);
__isl_give isl_union_set *isl_union_map_deltas(
    __isl_take isl_union_map *umap);

```

These functions return a (basic) set containing the differences between image elements and corresponding domain elements in the input.

```

__isl_give isl_basic_map *isl_basic_map_deltas_map(
    __isl_take isl_basic_map *bmap);
__isl_give isl_map *isl_map_deltas_map(
    __isl_take isl_map *map);
__isl_give isl_union_map *isl_union_map_deltas_map(
    __isl_take isl_union_map *umap);

```

The functions above construct a (basic, regular or union) relation that maps (a wrapped version of) the input relation to its delta set.

- Coalescing

Simplify the representation of a set or relation by trying to combine pairs of basic sets or relations into a single basic set or relation.

```

__isl_give isl_set *isl_set_coalesce(__isl_take isl_set *set);
__isl_give isl_map *isl_map_coalesce(__isl_take isl_map *map);
__isl_give isl_union_set *isl_union_set_coalesce(
    __isl_take isl_union_set *uset);
__isl_give isl_union_map *isl_union_map_coalesce(
    __isl_take isl_union_map *umap);

```

One of the methods for combining pairs of basic sets or relations can result in coefficients that are much larger than those that appear in the constraints of the input. By default, the coefficients are not allowed to grow larger, but this can be changed by unsetting the following option.

```
int isl_options_set_coalesce_bounded_wrapping(
    isl_ctx *ctx, int val);
int isl_options_get_coalesce_bounded_wrapping(
    isl_ctx *ctx);
```

- Detecting equalities

```
__isl_give isl_basic_set *isl_basic_set_detect_equalities(
    __isl_take isl_basic_set *bset);
__isl_give isl_basic_map *isl_basic_map_detect_equalities(
    __isl_take isl_basic_map *bmap);
__isl_give isl_set *isl_set_detect_equalities(
    __isl_take isl_set *set);
__isl_give isl_map *isl_map_detect_equalities(
    __isl_take isl_map *map);
__isl_give isl_union_set *isl_union_set_detect_equalities(
    __isl_take isl_union_set *uset);
__isl_give isl_union_map *isl_union_map_detect_equalities(
    __isl_take isl_union_map *umap);
```

Simplify the representation of a set or relation by detecting implicit equalities.

- Removing redundant constraints

```
__isl_give isl_basic_set *isl_basic_set_remove_redundancies(
    __isl_take isl_basic_set *bset);
__isl_give isl_set *isl_set_remove_redundancies(
    __isl_take isl_set *set);
__isl_give isl_basic_map *isl_basic_map_remove_redundancies(
    __isl_take isl_basic_map *bmap);
__isl_give isl_map *isl_map_remove_redundancies(
    __isl_take isl_map *map);
```

- Convex hull

```
__isl_give isl_basic_set *isl_set_convex_hull(
    __isl_take isl_set *set);
__isl_give isl_basic_map *isl_map_convex_hull(
    __isl_take isl_map *map);
```

If the input set or relation has any existentially quantified variables, then the result of these operations is currently undefined.

- Simple hull

```
#include <isl/set.h>
__isl_give isl_basic_set *
isl_set_unshifted_simple_hull(
    __isl_take isl_set *set);
__isl_give isl_basic_set *isl_set_simple_hull(
    __isl_take isl_set *set);
__isl_give isl_basic_set *
isl_set_unshifted_simple_hull_from_set_list(
    __isl_take isl_set *set,
    __isl_take isl_set_list *list);

#include <isl/map.h>
__isl_give isl_basic_map *
isl_map_unshifted_simple_hull(
    __isl_take isl_map *map);
__isl_give isl_basic_map *isl_map_simple_hull(
    __isl_take isl_map *map);

#include <isl/union_map.h>
__isl_give isl_union_map *isl_union_map_simple_hull(
    __isl_take isl_union_map *umap);
```

These functions compute a single basic set or relation that contains the whole input set or relation. In particular, the output is described by translates of the constraints describing the basic sets or relations in the input. In case of `isl_set_unshifted_simple_hull`, only the original constraints are used, without any translation. In case of `isl_set_unshifted_simple_hull_from_set_list`, the constraints are taken from the elements of the second argument.

(See Section 2.2.)

- Affine hull

```
__isl_give isl_basic_set *isl_basic_set_affine_hull(
    __isl_take isl_basic_set *bset);
__isl_give isl_basic_set *isl_set_affine_hull(
    __isl_take isl_set *set);
__isl_give isl_union_set *isl_union_set_affine_hull(
    __isl_take isl_union_set *uset);
__isl_give isl_basic_map *isl_basic_map_affine_hull(
    __isl_take isl_basic_map *bmap);
__isl_give isl_basic_map *isl_map_affine_hull(
    __isl_take isl_map *map);
__isl_give isl_union_map *isl_union_map_affine_hull(
    __isl_take isl_union_map *umap);
```

In case of union sets and relations, the affine hull is computed per space.

- Polyhedral hull

```
__isl_give isl_basic_set *isl_set_polyhedral_hull(
    __isl_take isl_set *set);
__isl_give isl_basic_map *isl_map_polyhedral_hull(
    __isl_take isl_map *map);
__isl_give isl_union_set *isl_union_set_polyhedral_hull(
    __isl_take isl_union_set *uset);
__isl_give isl_union_map *isl_union_map_polyhedral_hull(
    __isl_take isl_union_map *umap);
```

These functions compute a single basic set or relation not involving any existentially quantified variables that contains the whole input set or relation. In case of union sets and relations, the polyhedral hull is computed per space.

- Other approximations

```
__isl_give isl_basic_set *
isl_basic_set_drop_constraints_involving_dims(
    __isl_take isl_basic_set *bset,
    enum isl_dim_type type,
    unsigned first, unsigned n);
__isl_give isl_basic_map *
isl_basic_map_drop_constraints_involving_dims(
    __isl_take isl_basic_map *bmap,
    enum isl_dim_type type,
    unsigned first, unsigned n);
__isl_give isl_basic_set *
isl_basic_set_drop_constraints_not_involving_dims(
    __isl_take isl_basic_set *bset,
    enum isl_dim_type type,
    unsigned first, unsigned n);
__isl_give isl_set *
isl_set_drop_constraints_involving_dims(
    __isl_take isl_set *set,
    enum isl_dim_type type,
    unsigned first, unsigned n);
__isl_give isl_map *
isl_map_drop_constraints_involving_dims(
    __isl_take isl_map *map,
    enum isl_dim_type type,
    unsigned first, unsigned n);
```

These functions drop any constraints (not) involving the specified dimensions. Note that the result depends on the representation of the input.

- Feasibility

```

__isl_give isl_basic_set *isl_basic_set_sample(
    __isl_take isl_basic_set *bset);
__isl_give isl_basic_set *isl_set_sample(
    __isl_take isl_set *set);
__isl_give isl_basic_map *isl_basic_map_sample(
    __isl_take isl_basic_map *bmap);
__isl_give isl_basic_map *isl_map_sample(
    __isl_take isl_map *map);

```

If the input (basic) set or relation is non-empty, then return a singleton subset of the input. Otherwise, return an empty set.

- Optimization

```

#include <isl/ilp.h>
__isl_give isl_val *isl_basic_set_max_val(
    __isl_keep isl_basic_set *bset,
    __isl_keep isl_aff *obj);
__isl_give isl_val *isl_set_min_val(
    __isl_keep isl_set *set,
    __isl_keep isl_aff *obj);
__isl_give isl_val *isl_set_max_val(
    __isl_keep isl_set *set,
    __isl_keep isl_aff *obj);

```

Compute the minimum or maximum of the integer affine expression `obj` over the points in `set`, returning the result in `opt`. The result is `NULL` in case of an error, the optimal value in case there is one, negative infinity or infinity if the problem is unbounded and `NaN` if the problem is empty.

- Parametric optimization

```

__isl_give isl_pw_aff *isl_set_dim_min(
    __isl_take isl_set *set, int pos);
__isl_give isl_pw_aff *isl_set_dim_max(
    __isl_take isl_set *set, int pos);
__isl_give isl_pw_aff *isl_map_dim_max(
    __isl_take isl_map *map, int pos);

```

Compute the minimum or maximum of the given set or output dimension as a function of the parameters (and input dimensions), but independently of the other set or output dimensions. For lexicographic optimization, see §1.4.14.

- Dual

The following functions compute either the set of (rational) coefficient values of valid constraints for the given set or the set of (rational) values satisfying the constraints with coefficients from the given set. Internally, these two sets of functions perform essentially the same operations, except that the set of coefficients is assumed to be a cone, while the set of values may be any polyhedron. The current implementation is based on the Farkas lemma and Fourier-Motzkin elimination, but this may change or be made optional in future. In particular, future implementations may use different dualization algorithms or skip the elimination step.

```
__isl_give isl_basic_set *isl_basic_set_coefficients(
    __isl_take isl_basic_set *bset);
__isl_give isl_basic_set *isl_set_coefficients(
    __isl_take isl_set *set);
__isl_give isl_union_set *isl_union_set_coefficients(
    __isl_take isl_union_set *bset);
__isl_give isl_basic_set *isl_basic_set_solutions(
    __isl_take isl_basic_set *bset);
__isl_give isl_basic_set *isl_set_solutions(
    __isl_take isl_set *set);
__isl_give isl_union_set *isl_union_set_solutions(
    __isl_take isl_union_set *bset);
```

- Power

```
__isl_give isl_map *isl_map_fixed_power_val(
    __isl_take isl_map *map,
    __isl_take isl_val *exp);
__isl_give isl_union_map *
isl_union_map_fixed_power_val(
    __isl_take isl_union_map *umap,
    __isl_take isl_val *exp);
```

Compute the given power of map, where `exp` is assumed to be non-zero. If the exponent `exp` is negative, then the `-exp` th power of the inverse of map is computed.

```
__isl_give isl_map *isl_map_power(__isl_take isl_map *map,
    int *exact);
__isl_give isl_union_map *isl_union_map_power(
    __isl_take isl_union_map *umap, int *exact);
```

Compute a parametric representation for all positive powers k of map. The result maps k to a nested relation corresponding to the k th power of map. The result may be an overapproximation. If the result is known to be exact, then `*exact` is set to 1.

- Transitive closure

```
__isl_give isl_map *isl_map_transitive_closure(
    __isl_take isl_map *map, int *exact);
__isl_give isl_union_map *isl_union_map_transitive_closure(
    __isl_take isl_union_map *umap, int *exact);
```

Compute the transitive closure of `map`. The result may be an overapproximation. If the result is known to be exact, then `*exact` is set to 1.

- Reaching path lengths

```
__isl_give isl_map *isl_map_reaching_path_lengths(
    __isl_take isl_map *map, int *exact);
```

Compute a relation that maps each element in the range of `map` to the lengths of all paths composed of edges in `map` that end up in the given element. The result may be an overapproximation. If the result is known to be exact, then `*exact` is set to 1. To compute the *maximal* path length, the resulting relation should be postprocessed by `isl_map_lexmax`. In particular, if the input relation is a dependence relation (mapping sources to sinks), then the maximal path length corresponds to the free schedule. Note, however, that `isl_map_lexmax` expects the maximum to be finite, so if the path lengths are unbounded (possibly due to the overapproximation), then you will get an error message.

- Wrapping

```
#include <isl/space.h>
__isl_give isl_space *isl_space_wrap(
    __isl_take isl_space *space);
__isl_give isl_space *isl_space_unwrap(
    __isl_take isl_space *space);

#include <isl/set.h>
__isl_give isl_basic_map *isl_basic_set_unwrap(
    __isl_take isl_basic_set *bset);
__isl_give isl_map *isl_set_unwrap(
    __isl_take isl_set *set);

#include <isl/map.h>
__isl_give isl_basic_set *isl_basic_map_wrap(
    __isl_take isl_basic_map *bmap);
__isl_give isl_set *isl_map_wrap(
    __isl_take isl_map *map);

#include <isl/union_set.h>
__isl_give isl_union_map *isl_union_set_unwrap(
    __isl_take isl_union_set *uset);
```

```

#include <isl/union_map.h>
__isl_give isl_union_set *isl_union_map_wrap(
    __isl_take isl_union_map *umap);

```

The input to `isl_space_unwrap` should be the space of a set, while that of `isl_space_wrap` should be the space of a relation. Conversely, the output of `isl_space_unwrap` is the space of a relation, while that of `isl_space_wrap` is the space of a set.

- Flattening

Remove any internal structure of domain (and range) of the given set or relation. If there is any such internal structure in the input, then the name of the space is also removed.

```

#include <isl/local_space.h>
__isl_give isl_local_space *
isl_local_space_flatten_domain(
    __isl_take isl_local_space *ls);
__isl_give isl_local_space *
isl_local_space_flatten_range(
    __isl_take isl_local_space *ls);

#include <isl/set.h>
__isl_give isl_basic_set *isl_basic_set_flatten(
    __isl_take isl_basic_set *bset);
__isl_give isl_set *isl_set_flatten(
    __isl_take isl_set *set);

#include <isl/map.h>
__isl_give isl_basic_map *isl_basic_map_flatten_domain(
    __isl_take isl_basic_map *bmap);
__isl_give isl_basic_map *isl_basic_map_flatten_range(
    __isl_take isl_basic_map *bmap);
__isl_give isl_map *isl_map_flatten_range(
    __isl_take isl_map *map);
__isl_give isl_map *isl_map_flatten_domain(
    __isl_take isl_map *map);
__isl_give isl_basic_map *isl_basic_map_flatten(
    __isl_take isl_basic_map *bmap);
__isl_give isl_map *isl_map_flatten(
    __isl_take isl_map *map);

#include <isl/map.h>
__isl_give isl_map *isl_set_flatten_map(
    __isl_take isl_set *set);

```

The function above constructs a relation that maps the input set to a flattened version of the set.

- Lifting

Lift the input set to a space with extra dimensions corresponding to the existentially quantified variables in the input. In particular, the result lives in a wrapped map where the domain is the original space and the range corresponds to the original existentially quantified variables.

```
__isl_give isl_basic_set *isl_basic_set_lift(
    __isl_take isl_basic_set *bset);
__isl_give isl_set *isl_set_lift(
    __isl_take isl_set *set);
__isl_give isl_union_set *isl_union_set_lift(
    __isl_take isl_union_set *uset);
```

Given a local space that contains the existentially quantified variables of a set, a basic relation that, when applied to a basic set, has essentially the same effect as `isl_basic_set_lift`, can be constructed using the following function.

```
#include <isl/local_space.h>
__isl_give isl_basic_map *isl_local_space_lifting(
    __isl_take isl_local_space *ls);
```

- Internal Product

```
__isl_give isl_basic_map *isl_basic_map_zip(
    __isl_take isl_basic_map *bmap);
__isl_give isl_map *isl_map_zip(
    __isl_take isl_map *map);
__isl_give isl_union_map *isl_union_map_zip(
    __isl_take isl_union_map *umap);
```

Given a relation with nested relations for domain and range, interchange the range of the domain with the domain of the range.

- Currying

```
__isl_give isl_basic_map *isl_basic_map_curry(
    __isl_take isl_basic_map *bmap);
__isl_give isl_basic_map *isl_basic_map_uncurry(
    __isl_take isl_basic_map *bmap);
__isl_give isl_map *isl_map_curry(
    __isl_take isl_map *map);
__isl_give isl_map *isl_map_uncurry(
    __isl_take isl_map *map);
```

```

__isl_give isl_union_map *isl_union_map_curry(
    __isl_take isl_union_map *umap);
__isl_give isl_union_map *isl_union_map_uncurry(
    __isl_take isl_union_map *umap);

```

Given a relation with a nested relation for domain, the curry functions move the range of the nested relation out of the domain and use it as the domain of a nested relation in the range, with the original range as range of this nested relation. The uncurry functions perform the inverse operation.

- Aligning parameters

```

__isl_give isl_basic_set *isl_basic_set_align_params(
    __isl_take isl_basic_set *bset,
    __isl_take isl_space *model);
__isl_give isl_set *isl_set_align_params(
    __isl_take isl_set *set,
    __isl_take isl_space *model);
__isl_give isl_basic_map *isl_basic_map_align_params(
    __isl_take isl_basic_map *bmap,
    __isl_take isl_space *model);
__isl_give isl_map *isl_map_align_params(
    __isl_take isl_map *map,
    __isl_take isl_space *model);

```

Change the order of the parameters of the given set or relation such that the first parameters match those of model. This may involve the introduction of extra parameters. All parameters need to be named.

- Dimension manipulation

```

#include <isl/local_space.h>
__isl_give isl_local_space *isl_local_space_add_dims(
    __isl_take isl_local_space *ls,
    enum isl_dim_type type, unsigned n);
__isl_give isl_local_space *isl_local_space_insert_dims(
    __isl_take isl_local_space *ls,
    enum isl_dim_type type, unsigned first, unsigned n);
__isl_give isl_local_space *isl_local_space_drop_dims(
    __isl_take isl_local_space *ls,
    enum isl_dim_type type, unsigned first, unsigned n);

#include <isl/set.h>
__isl_give isl_basic_set *isl_basic_set_add_dims(
    __isl_take isl_basic_set *bset,
    enum isl_dim_type type, unsigned n);
__isl_give isl_set *isl_set_add_dims(

```

```

        __isl_take isl_set *set,
        enum isl_dim_type type, unsigned n);
__isl_give isl_basic_set *isl_basic_set_insert_dims(
        __isl_take isl_basic_set *bset,
        enum isl_dim_type type, unsigned pos,
        unsigned n);
__isl_give isl_set *isl_set_insert_dims(
        __isl_take isl_set *set,
        enum isl_dim_type type, unsigned pos, unsigned n);
__isl_give isl_basic_set *isl_basic_set_move_dims(
        __isl_take isl_basic_set *bset,
        enum isl_dim_type dst_type, unsigned dst_pos,
        enum isl_dim_type src_type, unsigned src_pos,
        unsigned n);
__isl_give isl_set *isl_set_move_dims(
        __isl_take isl_set *set,
        enum isl_dim_type dst_type, unsigned dst_pos,
        enum isl_dim_type src_type, unsigned src_pos,
        unsigned n);

#include <isl/map.h>
__isl_give isl_map *isl_map_add_dims(
        __isl_take isl_map *map,
        enum isl_dim_type type, unsigned n);
__isl_give isl_basic_map *isl_basic_map_insert_dims(
        __isl_take isl_basic_map *bmap,
        enum isl_dim_type type, unsigned pos,
        unsigned n);
__isl_give isl_map *isl_map_insert_dims(
        __isl_take isl_map *map,
        enum isl_dim_type type, unsigned pos, unsigned n);
__isl_give isl_basic_map *isl_basic_map_move_dims(
        __isl_take isl_basic_map *bmap,
        enum isl_dim_type dst_type, unsigned dst_pos,
        enum isl_dim_type src_type, unsigned src_pos,
        unsigned n);
__isl_give isl_map *isl_map_move_dims(
        __isl_take isl_map *map,
        enum isl_dim_type dst_type, unsigned dst_pos,
        enum isl_dim_type src_type, unsigned src_pos,
        unsigned n);

```

It is usually not advisable to directly change the (input or output) space of a set or a relation as this removes the name and the internal structure of the space. However, the above functions can be useful to add new parameters, assuming `isl_set_align_params` and `isl_map_align_params` are not sufficient.

1.4.14 Binary Operations

The two arguments of a binary operation not only need to live in the same `isl_ctx`, they currently also need to have the same (number of) parameters.

Basic Operations

- Intersection

```
#include <isl/local_space.h>
__isl_give isl_local_space *isl_local_space_intersect(
    __isl_take isl_local_space *ls1,
    __isl_take isl_local_space *ls2);

#include <isl/set.h>
__isl_give isl_basic_set *isl_basic_set_intersect_params(
    __isl_take isl_basic_set *bset1,
    __isl_take isl_basic_set *bset2);
__isl_give isl_basic_set *isl_basic_set_intersect(
    __isl_take isl_basic_set *bset1,
    __isl_take isl_basic_set *bset2);
__isl_give isl_basic_set *isl_basic_set_list_intersect(
    __isl_take struct isl_basic_set_list *list);
__isl_give isl_set *isl_set_intersect_params(
    __isl_take isl_set *set,
    __isl_take isl_set *params);
__isl_give isl_set *isl_set_intersect(
    __isl_take isl_set *set1,
    __isl_take isl_set *set2);

#include <isl/map.h>
__isl_give isl_basic_map *isl_basic_map_intersect_domain(
    __isl_take isl_basic_map *bmap,
    __isl_take isl_basic_set *bset);
__isl_give isl_basic_map *isl_basic_map_intersect_range(
    __isl_take isl_basic_map *bmap,
    __isl_take isl_basic_set *bset);
__isl_give isl_basic_map *isl_basic_map_intersect(
    __isl_take isl_basic_map *bmap1,
    __isl_take isl_basic_map *bmap2);
__isl_give isl_map *isl_map_intersect_params(
    __isl_take isl_map *map,
    __isl_take isl_set *params);
__isl_give isl_map *isl_map_intersect_domain(
    __isl_take isl_map *map,
    __isl_take isl_set *set);
```

```

__isl_give isl_map *isl_map_intersect_range(
    __isl_take isl_map *map,
    __isl_take isl_set *set);
__isl_give isl_map *isl_map_intersect(
    __isl_take isl_map *map1,
    __isl_take isl_map *map2);

#include <isl/union_set.h>
__isl_give isl_union_set *isl_union_set_intersect_params(
    __isl_take isl_union_set *uset,
    __isl_take isl_set *set);
__isl_give isl_union_set *isl_union_set_intersect(
    __isl_take isl_union_set *uset1,
    __isl_take isl_union_set *uset2);

#include <isl/union_map.h>
__isl_give isl_union_map *isl_union_map_intersect_params(
    __isl_take isl_union_map *umap,
    __isl_take isl_set *set);
__isl_give isl_union_map *isl_union_map_intersect_domain(
    __isl_take isl_union_map *umap,
    __isl_take isl_union_set *uset);
__isl_give isl_union_map *isl_union_map_intersect_range(
    __isl_take isl_union_map *umap,
    __isl_take isl_union_set *uset);
__isl_give isl_union_map *isl_union_map_intersect(
    __isl_take isl_union_map *umap1,
    __isl_take isl_union_map *umap2);

```

The second argument to the `_params` functions needs to be a parametric (basic) set. For the other functions, a parametric set for either argument is only allowed if the other argument is a parametric set as well. The list passed to `isl_basic_set_list_intersect` needs to have at least one element and all elements need to live in the same space.

- Union

```

__isl_give isl_set *isl_basic_set_union(
    __isl_take isl_basic_set *bset1,
    __isl_take isl_basic_set *bset2);
__isl_give isl_map *isl_basic_map_union(
    __isl_take isl_basic_map *bmap1,
    __isl_take isl_basic_map *bmap2);
__isl_give isl_set *isl_set_union(
    __isl_take isl_set *set1,
    __isl_take isl_set *set2);

```

```

__isl_give isl_map *isl_map_union(
    __isl_take isl_map *map1,
    __isl_take isl_map *map2);
__isl_give isl_union_set *isl_union_set_union(
    __isl_take isl_union_set *uset1,
    __isl_take isl_union_set *uset2);
__isl_give isl_union_map *isl_union_map_union(
    __isl_take isl_union_map *umap1,
    __isl_take isl_union_map *umap2);

```

- Set difference

```

__isl_give isl_set *isl_set_subtract(
    __isl_take isl_set *set1,
    __isl_take isl_set *set2);
__isl_give isl_map *isl_map_subtract(
    __isl_take isl_map *map1,
    __isl_take isl_map *map2);
__isl_give isl_map *isl_map_subtract_domain(
    __isl_take isl_map *map,
    __isl_take isl_set *dom);
__isl_give isl_map *isl_map_subtract_range(
    __isl_take isl_map *map,
    __isl_take isl_set *dom);
__isl_give isl_union_set *isl_union_set_subtract(
    __isl_take isl_union_set *uset1,
    __isl_take isl_union_set *uset2);
__isl_give isl_union_map *isl_union_map_subtract(
    __isl_take isl_union_map *umap1,
    __isl_take isl_union_map *umap2);
__isl_give isl_union_map *isl_union_map_subtract_domain(
    __isl_take isl_union_map *umap,
    __isl_take isl_union_set *dom);
__isl_give isl_union_map *isl_union_map_subtract_range(
    __isl_take isl_union_map *umap,
    __isl_take isl_union_set *dom);

```

- Application

```

__isl_give isl_basic_set *isl_basic_set_apply(
    __isl_take isl_basic_set *bset,
    __isl_take isl_basic_map *bmap);
__isl_give isl_set *isl_set_apply(
    __isl_take isl_set *set,
    __isl_take isl_map *map);
__isl_give isl_union_set *isl_union_set_apply(

```



```

        __isl_take isl_union_set *uset,
        __isl_take isl_union_map *umap);
__isl_give isl_basic_map *isl_basic_map_apply_domain(
        __isl_take isl_basic_map *bmap1,
        __isl_take isl_basic_map *bmap2);
__isl_give isl_basic_map *isl_basic_map_apply_range(
        __isl_take isl_basic_map *bmap1,
        __isl_take isl_basic_map *bmap2);
__isl_give isl_map *isl_map_apply_domain(
        __isl_take isl_map *map1,
        __isl_take isl_map *map2);
__isl_give isl_union_map *isl_union_map_apply_domain(
        __isl_take isl_union_map *umap1,
        __isl_take isl_union_map *umap2);
__isl_give isl_map *isl_map_apply_range(
        __isl_take isl_map *map1,
        __isl_take isl_map *map2);
__isl_give isl_union_map *isl_union_map_apply_range(
        __isl_take isl_union_map *umap1,
        __isl_take isl_union_map *umap2);

```

- Preimage

```

#include <isl/set.h>
__isl_give isl_basic_set *
isl_basic_set_preimage_multi_aff(
        __isl_take isl_basic_set *bset,
        __isl_take isl_multi_aff *ma);
__isl_give isl_set *isl_set_preimage_multi_aff(
        __isl_take isl_set *set,
        __isl_take isl_multi_aff *ma);
__isl_give isl_set *isl_set_preimage_pw_multi_aff(
        __isl_take isl_set *set,
        __isl_take isl_pw_multi_aff *pma);
__isl_give isl_set *isl_set_preimage_multi_pw_aff(
        __isl_take isl_set *set,
        __isl_take isl_multi_pw_aff *mpa);

#include <isl/union_set.h>
__isl_give isl_union_set *
isl_union_set_preimage_multi_aff(
        __isl_take isl_union_set *uset,
        __isl_take isl_multi_aff *ma);
__isl_give isl_union_set *
isl_union_set_preimage_pw_multi_aff(
        __isl_take isl_union_set *uset,

```

```

        __isl_take isl_pw_multi_aff *pma);
__isl_give isl_union_set *
isl_union_set_preimage_union_pw_multi_aff(
        __isl_take isl_union_set *uset,
        __isl_take isl_union_pw_multi_aff *upma);

#include <isl/map.h>
__isl_give isl_basic_map *
isl_basic_map_preimage_domain_multi_aff(
        __isl_take isl_basic_map *bmap,
        __isl_take isl_multi_aff *ma);
__isl_give isl_map *isl_map_preimage_domain_multi_aff(
        __isl_take isl_map *map,
        __isl_take isl_multi_aff *ma);
__isl_give isl_map *isl_map_preimage_range_multi_aff(
        __isl_take isl_map *map,
        __isl_take isl_multi_aff *ma);
__isl_give isl_map *
isl_map_preimage_domain_pw_multi_aff(
        __isl_take isl_map *map,
        __isl_take isl_pw_multi_aff *pma);
__isl_give isl_map *
isl_map_preimage_range_pw_multi_aff(
        __isl_take isl_map *map,
        __isl_take isl_pw_multi_aff *pma);
__isl_give isl_map *
isl_map_preimage_domain_multi_pw_aff(
        __isl_take isl_map *map,
        __isl_take isl_multi_pw_aff *mpa);
__isl_give isl_basic_map *
isl_basic_map_preimage_range_multi_aff(
        __isl_take isl_basic_map *bmap,
        __isl_take isl_multi_aff *ma);

#include <isl/union_map.h>
__isl_give isl_union_map *
isl_union_map_preimage_domain_multi_aff(
        __isl_take isl_union_map *umap,
        __isl_take isl_multi_aff *ma);
__isl_give isl_union_map *
isl_union_map_preimage_range_multi_aff(
        __isl_take isl_union_map *umap,
        __isl_take isl_multi_aff *ma);
__isl_give isl_union_map *
isl_union_map_preimage_domain_pw_multi_aff(
        __isl_take isl_union_map *umap,

```

```

        __isl_take isl_pw_multi_aff *pma);
__isl_give isl_union_map *
isl_union_map_preimage_range_pw_multi_aff(
    __isl_take isl_union_map *umap,
    __isl_take isl_pw_multi_aff *pma);
__isl_give isl_union_map *
isl_union_map_preimage_domain_union_pw_multi_aff(
    __isl_take isl_union_map *umap,
    __isl_take isl_union_pw_multi_aff *upma);
__isl_give isl_union_map *
isl_union_map_preimage_range_union_pw_multi_aff(
    __isl_take isl_union_map *umap,
    __isl_take isl_union_pw_multi_aff *upma);

```

These functions compute the preimage of the given set or map domain/range under the given function. In other words, the expression is plugged into the set description or into the domain/range of the map. Objects of types `isl_multi_aff` and `isl_pw_multi_aff` are described in §1.4.21.

- Cartesian Product

```

#include <isl/space.h>
__isl_give isl_space *isl_space_product(
    __isl_take isl_space *space1,
    __isl_take isl_space *space2);
__isl_give isl_space *isl_space_domain_product(
    __isl_take isl_space *space1,
    __isl_take isl_space *space2);
__isl_give isl_space *isl_space_range_product(
    __isl_take isl_space *space1,
    __isl_take isl_space *space2);

```

The functions `isl_space_product`, `isl_space_domain_product` and `isl_space_range_product` take pairs or relation spaces and produce a single relations space, where either the domain, the range or both domain and range are wrapped spaces of relations between the domains and/or ranges of the input spaces. If the product is only constructed over the domain or the range then the ranges or the domains of the inputs should be the same. The function `isl_space_product` also accepts a pair of set spaces, in which case it returns a wrapped space of a relation between the two input spaces.

```

#include <isl/set.h>
__isl_give isl_set *isl_set_product(
    __isl_take isl_set *set1,
    __isl_take isl_set *set2);

```

```

#include <isl/map.h>
__isl_give isl_basic_map *isl_basic_map_domain_product(
    __isl_take isl_basic_map *bmap1,
    __isl_take isl_basic_map *bmap2);
__isl_give isl_basic_map *isl_basic_map_range_product(
    __isl_take isl_basic_map *bmap1,
    __isl_take isl_basic_map *bmap2);
__isl_give isl_basic_map *isl_basic_map_product(
    __isl_take isl_basic_map *bmap1,
    __isl_take isl_basic_map *bmap2);
__isl_give isl_map *isl_map_domain_product(
    __isl_take isl_map *map1,
    __isl_take isl_map *map2);
__isl_give isl_map *isl_map_range_product(
    __isl_take isl_map *map1,
    __isl_take isl_map *map2);
__isl_give isl_map *isl_map_product(
    __isl_take isl_map *map1,
    __isl_take isl_map *map2);

#include <isl/union_set.h>
__isl_give isl_union_set *isl_union_set_product(
    __isl_take isl_union_set *uset1,
    __isl_take isl_union_set *uset2);

#include <isl/union_map.h>
__isl_give isl_union_map *isl_union_map_domain_product(
    __isl_take isl_union_map *umap1,
    __isl_take isl_union_map *umap2);
__isl_give isl_union_map *isl_union_map_range_product(
    __isl_take isl_union_map *umap1,
    __isl_take isl_union_map *umap2);
__isl_give isl_union_map *isl_union_map_product(
    __isl_take isl_union_map *umap1,
    __isl_take isl_union_map *umap2);

```

The above functions compute the cross product of the given sets or relations. The domains and ranges of the results are wrapped maps between domains and ranges of the inputs. To obtain a “flat” product, use the following functions instead.

```

__isl_give isl_basic_set *isl_basic_set_flat_product(
    __isl_take isl_basic_set *bset1,
    __isl_take isl_basic_set *bset2);
__isl_give isl_set *isl_set_flat_product(
    __isl_take isl_set *set1,

```

```

        __isl_take isl_set *set2);
__isl_give isl_basic_map *isl_basic_map_flat_range_product(
        __isl_take isl_basic_map *bmap1,
        __isl_take isl_basic_map *bmap2);
__isl_give isl_map *isl_map_flat_domain_product(
        __isl_take isl_map *map1,
        __isl_take isl_map *map2);
__isl_give isl_map *isl_map_flat_range_product(
        __isl_take isl_map *map1,
        __isl_take isl_map *map2);
__isl_give isl_union_map *isl_union_map_flat_range_product(
        __isl_take isl_union_map *umap1,
        __isl_take isl_union_map *umap2);
__isl_give isl_basic_map *isl_basic_map_flat_product(
        __isl_take isl_basic_map *bmap1,
        __isl_take isl_basic_map *bmap2);
__isl_give isl_map *isl_map_flat_product(
        __isl_take isl_map *map1,
        __isl_take isl_map *map2);

#include <isl/space.h>
__isl_give isl_space *isl_space_domain_factor_domain(
        __isl_take isl_space *space);
__isl_give isl_space *isl_space_range_factor_domain(
        __isl_take isl_space *space);
__isl_give isl_space *isl_space_range_factor_range(
        __isl_take isl_space *space);

```

The functions `isl_space_range_factor_domain` and `isl_space_range_factor_range` extract the two arguments from the result of a call to `isl_space_range_product`.

The arguments of a call to `isl_map_range_product` can be extracted from the result using the following two functions.

```

#include <isl/map.h>
__isl_give isl_map *isl_map_range_factor_domain(
        __isl_take isl_map *map);
__isl_give isl_map *isl_map_range_factor_range(
        __isl_take isl_map *map);

```

- Simplification

```

__isl_give isl_basic_set *isl_basic_set_gist(
        __isl_take isl_basic_set *bset,
        __isl_take isl_basic_set *context);
__isl_give isl_set *isl_set_gist(__isl_take isl_set *set,

```

```

        __isl_take isl_set *context);
__isl_give isl_set *isl_set_gist_params(
        __isl_take isl_set *set,
        __isl_take isl_set *context);
__isl_give isl_union_set *isl_union_set_gist(
        __isl_take isl_union_set *uset,
        __isl_take isl_union_set *context);
__isl_give isl_union_set *isl_union_set_gist_params(
        __isl_take isl_union_set *uset,
        __isl_take isl_set *set);
__isl_give isl_basic_map *isl_basic_map_gist(
        __isl_take isl_basic_map *bmap,
        __isl_take isl_basic_map *context);
__isl_give isl_map *isl_map_gist(__isl_take isl_map *map,
        __isl_take isl_map *context);
__isl_give isl_map *isl_map_gist_params(
        __isl_take isl_map *map,
        __isl_take isl_set *context);
__isl_give isl_map *isl_map_gist_domain(
        __isl_take isl_map *map,
        __isl_take isl_set *context);
__isl_give isl_map *isl_map_gist_range(
        __isl_take isl_map *map,
        __isl_take isl_set *context);
__isl_give isl_union_map *isl_union_map_gist(
        __isl_take isl_union_map *umap,
        __isl_take isl_union_map *context);
__isl_give isl_union_map *isl_union_map_gist_params(
        __isl_take isl_union_map *umap,
        __isl_take isl_set *set);
__isl_give isl_union_map *isl_union_map_gist_domain(
        __isl_take isl_union_map *umap,
        __isl_take isl_union_set *uset);
__isl_give isl_union_map *isl_union_map_gist_range(
        __isl_take isl_union_map *umap,
        __isl_take isl_union_set *uset);

```

The gist operation returns a set or relation that has the same intersection with the context as the input set or relation. Any implicit equality in the intersection is made explicit in the result, while all inequalities that are redundant with respect to the intersection are removed. In case of union sets and relations, the gist operation is performed per space.

Lexicographic Optimization

Given a (basic) set `set` (or `bset`) and a zero-dimensional domain `dom`, the following functions compute a set that contains the lexicographic minimum or maximum of the elements in `set` (or `bset`) for those values of the parameters that satisfy `dom`. If `empty` is not NULL, then `*empty` is assigned a set that contains the parameter values in `dom` for which `set` (or `bset`) has no elements. In other words, the union of the parameter values for which the result is non-empty and of `*empty` is equal to `dom`.

```
__isl_give isl_set *isl_basic_set_partial_lexmin(
    __isl_take isl_basic_set *bset,
    __isl_take isl_basic_set *dom,
    __isl_give isl_set **empty);
__isl_give isl_set *isl_basic_set_partial_lexmax(
    __isl_take isl_basic_set *bset,
    __isl_take isl_basic_set *dom,
    __isl_give isl_set **empty);
__isl_give isl_set *isl_set_partial_lexmin(
    __isl_take isl_set *set, __isl_take isl_set *dom,
    __isl_give isl_set **empty);
__isl_give isl_set *isl_set_partial_lexmax(
    __isl_take isl_set *set, __isl_take isl_set *dom,
    __isl_give isl_set **empty);
```

Given a (basic) set `set` (or `bset`), the following functions simply return a set containing the lexicographic minimum or maximum of the elements in `set` (or `bset`). In case of union sets, the optimum is computed per space.

```
__isl_give isl_set *isl_basic_set_lexmin(
    __isl_take isl_basic_set *bset);
__isl_give isl_set *isl_basic_set_lexmax(
    __isl_take isl_basic_set *bset);
__isl_give isl_set *isl_set_lexmin(
    __isl_take isl_set *set);
__isl_give isl_set *isl_set_lexmax(
    __isl_take isl_set *set);
__isl_give isl_union_set *isl_union_set_lexmin(
    __isl_take isl_union_set *uset);
__isl_give isl_union_set *isl_union_set_lexmax(
    __isl_take isl_union_set *uset);
```

Given a (basic) relation map (or `bmap`) and a domain `dom`, the following functions compute a relation that maps each element of `dom` to the single lexicographic minimum or maximum of the elements that are associated to that same element in `map` (or `bmap`). If `empty` is not NULL, then `*empty` is assigned a set that contains the elements in `dom` that do not map to any elements in `map` (or `bmap`). In other words, the union of the domain of the result and of `*empty` is equal to `dom`.

```

__isl_give isl_map *isl_basic_map_partial_lexmax(
    __isl_take isl_basic_map *bmap,
    __isl_take isl_basic_set *dom,
    __isl_give isl_set **empty);
__isl_give isl_map *isl_basic_map_partial_lexmin(
    __isl_take isl_basic_map *bmap,
    __isl_take isl_basic_set *dom,
    __isl_give isl_set **empty);
__isl_give isl_map *isl_map_partial_lexmax(
    __isl_take isl_map *map, __isl_take isl_set *dom,
    __isl_give isl_set **empty);
__isl_give isl_map *isl_map_partial_lexmin(
    __isl_take isl_map *map, __isl_take isl_set *dom,
    __isl_give isl_set **empty);

```

Given a (basic) map `map` (or `bmap`), the following functions simply return a map mapping each element in the domain of `map` (or `bmap`) to the lexicographic minimum or maximum of all elements associated to that element. In case of union relations, the optimum is computed per space.

```

__isl_give isl_map *isl_basic_map_lexmin(
    __isl_take isl_basic_map *bmap);
__isl_give isl_map *isl_basic_map_lexmax(
    __isl_take isl_basic_map *bmap);
__isl_give isl_map *isl_map_lexmin(
    __isl_take isl_map *map);
__isl_give isl_map *isl_map_lexmax(
    __isl_take isl_map *map);
__isl_give isl_union_map *isl_union_map_lexmin(
    __isl_take isl_union_map *umap);
__isl_give isl_union_map *isl_union_map_lexmax(
    __isl_take isl_union_map *umap);

```

The following functions return their result in the form of a piecewise multi-affine expression (See §1.4.21), but are otherwise equivalent to the corresponding functions returning a basic set or relation.

```

__isl_give isl_pw_multi_aff *
isl_basic_map_lexmin_pw_multi_aff(
    __isl_take isl_basic_map *bmap);
__isl_give isl_pw_multi_aff *
isl_basic_set_partial_lexmin_pw_multi_aff(
    __isl_take isl_basic_set *bset,
    __isl_take isl_basic_set *dom,
    __isl_give isl_set **empty);
__isl_give isl_pw_multi_aff *
isl_basic_set_partial_lexmax_pw_multi_aff(

```



```

        __isl_take isl_basic_set *bset,
        __isl_take isl_basic_set *dom,
        __isl_give isl_set **empty);
__isl_give isl_pw_multi_aff *
isl_basic_map_partial_lexmin_pw_multi_aff(
    __isl_take isl_basic_map *bmap,
    __isl_take isl_basic_set *dom,
    __isl_give isl_set **empty);
__isl_give isl_pw_multi_aff *
isl_basic_map_partial_lexmax_pw_multi_aff(
    __isl_take isl_basic_map *bmap,
    __isl_take isl_basic_set *dom,
    __isl_give isl_set **empty);
__isl_give isl_pw_multi_aff *isl_set_lexmin_pw_multi_aff(
    __isl_take isl_set *set);
__isl_give isl_pw_multi_aff *isl_set_lexmax_pw_multi_aff(
    __isl_take isl_set *set);
__isl_give isl_pw_multi_aff *isl_map_lexmin_pw_multi_aff(
    __isl_take isl_map *map);
__isl_give isl_pw_multi_aff *isl_map_lexmax_pw_multi_aff(
    __isl_take isl_map *map);

```

1.4.15 Lists

Lists are defined over several element types, including `isl_val`, `isl_id`, `isl_aff`, `isl_pw_aff`, `isl_constraint`, `isl_basic_set`, `isl_set`, `isl_ast_expr` and `isl_ast_node`. Here we take lists of `isl_sets` as an example. Lists can be created, copied, modified and freed using the following functions.

```

#include <isl/set.h>
__isl_give isl_set_list *isl_set_list_from_set(
    __isl_take isl_set *el);
__isl_give isl_set_list *isl_set_list_alloc(
    isl_ctx *ctx, int n);
__isl_give isl_set_list *isl_set_list_copy(
    __isl_keep isl_set_list *list);
__isl_give isl_set_list *isl_set_list_insert(
    __isl_take isl_set_list *list, unsigned pos,
    __isl_take isl_set *el);
__isl_give isl_set_list *isl_set_list_add(
    __isl_take isl_set_list *list,
    __isl_take isl_set *el);
__isl_give isl_set_list *isl_set_list_drop(
    __isl_take isl_set_list *list,
    unsigned first, unsigned n);
__isl_give isl_set_list *isl_set_list_set_set(

```

```

__isl_take isl_set_list *list, int index,
__isl_take isl_set *set);
__isl_give isl_set_list *isl_set_list_concat(
__isl_take isl_set_list *list1,
__isl_take isl_set_list *list2);
__isl_give isl_set_list *isl_set_list_sort(
__isl_take isl_set_list *list,
int (*cmp)(__isl_keep isl_set *a,
__isl_keep isl_set *b, void *user),
void *user);
__isl_null isl_set_list *isl_set_list_free(
__isl_take isl_set_list *list);

```

`isl_set_list_alloc` creates an empty list with a capacity for `n` elements. `isl_set_list_from_set` creates a list with a single element.

Lists can be inspected using the following functions.

```

#include <isl/set.h>
isl_ctx *isl_set_list_get_ctx(__isl_keep isl_set_list *list);
int isl_set_list_n_set(__isl_keep isl_set_list *list);
__isl_give isl_set *isl_set_list_get_set(
__isl_keep isl_set_list *list, int index);
int isl_set_list_foreach(__isl_keep isl_set_list *list,
int (*fn)(__isl_take isl_set *el, void *user),
void *user);
int isl_set_list_foreach_scc(__isl_keep isl_set_list *list,
int (*follows)(__isl_keep isl_set *a,
__isl_keep isl_set *b, void *user),
void *follows_user
int (*fn)(__isl_take isl_set *el, void *user),
void *fn_user);

```

The function `isl_set_list_foreach_scc` calls `fn` on each of the strongly connected components of the graph with as vertices the elements of `list` and a directed edge from vertex `b` to vertex `a` iff `follows(a, b)` returns 1. The callback `follows` and `fn` should return -1 on error.

Lists can be printed using

```

#include <isl/set.h>
__isl_give isl_printer *isl_printer_print_set_list(
__isl_take isl_printer *p,
__isl_keep isl_set_list *list);

```

1.4.16 Associative arrays

Associative arrays map isl objects of a specific type to isl objects of some (other) specific type. They are defined for several pairs of types, including (`isl_map`, `isl_basic_set`),

(isl_id, isl_ast_expr) and (isl_id, isl_pw.aff). Here, we take associative arrays that map isl_ids to isl_ast_exprs as an example.

Associative arrays can be created, copied and freed using the following functions.

```
#include <isl/id_to_ast_expr.h>
__isl_give id_to_ast_expr *isl_id_to_ast_expr_alloc(
    isl_ctx *ctx, int min_size);
__isl_give id_to_ast_expr *isl_id_to_ast_expr_copy(
    __isl_keep id_to_ast_expr *id2expr);
__isl_null id_to_ast_expr *isl_id_to_ast_expr_free(
    __isl_take id_to_ast_expr *id2expr);
```

The min_size argument to isl_id_to_ast_expr_alloc can be used to specify the expected size of the associative array. The associative array will be grown automatically as needed.

Associative arrays can be inspected using the following functions.

```
#include <isl/id_to_ast_expr.h>
isl_ctx *isl_id_to_ast_expr_get_ctx(
    __isl_keep id_to_ast_expr *id2expr);
int isl_id_to_ast_expr_has(
    __isl_keep id_to_ast_expr *id2expr,
    __isl_keep isl_id *key);
__isl_give isl_ast_expr *isl_id_to_ast_expr_get(
    __isl_keep id_to_ast_expr *id2expr,
    __isl_take isl_id *key);
int isl_id_to_ast_expr_foreach(
    __isl_keep id_to_ast_expr *id2expr,
    int (*fn)(__isl_take isl_id *key,
              __isl_take isl_ast_expr *val, void *user),
    void *user);
```

They can be modified using the following function.

```
#include <isl/id_to_ast_expr.h>
__isl_give id_to_ast_expr *isl_id_to_ast_expr_set(
    __isl_take id_to_ast_expr *id2expr,
    __isl_take isl_id *key,
    __isl_take isl_ast_expr *val);
__isl_give id_to_ast_expr *isl_id_to_ast_expr_drop(
    __isl_take id_to_ast_expr *id2expr,
    __isl_take isl_id *key);
```

Associative arrays can be printed using the following function.

```
#include <isl/id_to_ast_expr.h>
__isl_give isl_printer *isl_printer_print_id_to_ast_expr(
    __isl_take isl_printer *p,
    __isl_keep id_to_ast_expr *id2expr);
```

1.4.17 Multiple Values

An `isl_multi_val` object represents a sequence of zero or more values, living in a set space.

An `isl_multi_val` can be constructed from an `isl_val_list` using the following function

```
#include <isl/val.h>
__isl_give isl_multi_val *isl_multi_val_from_val_list(
    __isl_take isl_space *space,
    __isl_take isl_val_list *list);
```

The zero multiple value (with value zero for each set dimension) can be created using the following function.

```
#include <isl/val.h>
__isl_give isl_multi_val *isl_multi_val_zero(
    __isl_take isl_space *space);
```

Multiple values can be copied and freed using

```
#include <isl/val.h>
__isl_give isl_multi_val *isl_multi_val_copy(
    __isl_keep isl_multi_val *mv);
__isl_null isl_multi_val *isl_multi_val_free(
    __isl_take isl_multi_val *mv);
```

They can be inspected using

```
#include <isl/val.h>
isl_ctx *isl_multi_val_get_ctx(
    __isl_keep isl_multi_val *mv);
unsigned isl_multi_val_dim(__isl_keep isl_multi_val *mv,
    enum isl_dim_type type);
__isl_give isl_val *isl_multi_val_get_val(
    __isl_keep isl_multi_val *mv, int pos);
int isl_multi_val_find_dim_by_id(
    __isl_keep isl_multi_val *mv,
    enum isl_dim_type type, __isl_keep isl_id *id);
__isl_give isl_id *isl_multi_val_get_dim_id(
    __isl_keep isl_multi_val *mv,
    enum isl_dim_type type, unsigned pos);
const char *isl_multi_val_get_tuple_name(
    __isl_keep isl_multi_val *mv,
    enum isl_dim_type type);
int isl_multi_val_has_tuple_id(__isl_keep isl_multi_val *mv,
    enum isl_dim_type type);
__isl_give isl_id *isl_multi_val_get_tuple_id(
```

```

        __isl_keep isl_multi_val *mv,
        enum isl_dim_type type);
int isl_multi_val_range_is_wrapping(
    __isl_keep isl_multi_val *mv);

```

They can be modified using

```

#include <isl/val.h>
__isl_give isl_multi_val *isl_multi_val_set_val(
    __isl_take isl_multi_val *mv, int pos,
    __isl_take isl_val *val);
__isl_give isl_multi_val *isl_multi_val_set_dim_name(
    __isl_take isl_multi_val *mv,
    enum isl_dim_type type, unsigned pos, const char *s);
__isl_give isl_multi_val *isl_multi_val_set_dim_id(
    __isl_take isl_multi_val *mv,
    enum isl_dim_type type, unsigned pos,
    __isl_take isl_id *id);
__isl_give isl_multi_val *isl_multi_val_set_tuple_name(
    __isl_take isl_multi_val *mv,
    enum isl_dim_type type, const char *s);
__isl_give isl_multi_val *isl_multi_val_set_tuple_id(
    __isl_take isl_multi_val *mv,
    enum isl_dim_type type, __isl_take isl_id *id);
__isl_give isl_multi_val *isl_multi_val_reset_tuple_id(
    __isl_take isl_multi_val *mv,
    enum isl_dim_type type);
__isl_give isl_multi_val *isl_multi_val_reset_user(
    __isl_take isl_multi_val *mv);

__isl_give isl_multi_val *isl_multi_val_insert_dims(
    __isl_take isl_multi_val *mv,
    enum isl_dim_type type, unsigned first, unsigned n);
__isl_give isl_multi_val *isl_multi_val_add_dims(
    __isl_take isl_multi_val *mv,
    enum isl_dim_type type, unsigned n);
__isl_give isl_multi_val *isl_multi_val_drop_dims(
    __isl_take isl_multi_val *mv,
    enum isl_dim_type type, unsigned first, unsigned n);

```

Operations include

```

#include <isl/val.h>
__isl_give isl_multi_val *isl_multi_val_align_params(
    __isl_take isl_multi_val *mv,
    __isl_take isl_space *model);
__isl_give isl_multi_val *isl_multi_val_from_range(

```

```

        __isl_take isl_multi_val *mv);
__isl_give isl_multi_val *isl_multi_val_range_splice(
        __isl_take isl_multi_val *mv1, unsigned pos,
        __isl_take isl_multi_val *mv2);
__isl_give isl_multi_val *isl_multi_val_range_product(
        __isl_take isl_multi_val *mv1,
        __isl_take isl_multi_val *mv2);
__isl_give isl_multi_val *
isl_multi_val_range_factor_domain(
        __isl_take isl_multi_val *mv);
__isl_give isl_multi_val *
isl_multi_val_range_factor_range(
        __isl_take isl_multi_val *mv);
__isl_give isl_multi_val *isl_multi_val_flat_range_product(
        __isl_take isl_multi_val *mv1,
        __isl_take isl_multi_val *mv2);
__isl_give isl_multi_val *isl_multi_val_product(
        __isl_take isl_multi_val *mv1,
        __isl_take isl_multi_val *mv2);
__isl_give isl_multi_val *isl_multi_val_add_val(
        __isl_take isl_multi_val *mv,
        __isl_take isl_val *v);
__isl_give isl_multi_val *isl_multi_val_mod_val(
        __isl_take isl_multi_val *mv,
        __isl_take isl_val *v);
__isl_give isl_multi_val *isl_multi_val_scale_val(
        __isl_take isl_multi_val *mv,
        __isl_take isl_val *v);
__isl_give isl_multi_val *isl_multi_val_scale_multi_val(
        __isl_take isl_multi_val *mv1,
        __isl_take isl_multi_val *mv2);
__isl_give isl_multi_val *
isl_multi_val_scale_down_multi_val(
        __isl_take isl_multi_val *mv1,
        __isl_take isl_multi_val *mv2);

```

A multiple value can be printed using

```

__isl_give isl_printer *isl_printer_print_multi_val(
        __isl_take isl_printer *p,
        __isl_keep isl_multi_val *mv);

```

1.4.18 Vectors

Vectors can be created, copied and freed using the following functions.

```
#include <isl/vec.h>
```

```

__isl_give isl_vec *isl_vec_alloc(isl_ctx *ctx,
    unsigned size);
__isl_give isl_vec *isl_vec_copy(__isl_keep isl_vec *vec);
__isl_null isl_vec *isl_vec_free(__isl_take isl_vec *vec);

```

Note that the elements of a newly created vector may have arbitrary values. The elements can be changed and inspected using the following functions.

```

isl_ctx *isl_vec_get_ctx(__isl_keep isl_vec *vec);
int isl_vec_size(__isl_keep isl_vec *vec);
__isl_give isl_val *isl_vec_get_element_val(
    __isl_keep isl_vec *vec, int pos);
__isl_give isl_vec *isl_vec_set_element_si(
    __isl_take isl_vec *vec, int pos, int v);
__isl_give isl_vec *isl_vec_set_element_val(
    __isl_take isl_vec *vec, int pos,
    __isl_take isl_val *v);
__isl_give isl_vec *isl_vec_set_si(__isl_take isl_vec *vec,
    int v);
__isl_give isl_vec *isl_vec_set_val(
    __isl_take isl_vec *vec, __isl_take isl_val *v);
int isl_vec_cmp_element(__isl_keep isl_vec *vec1,
    __isl_keep isl_vec *vec2, int pos);

```

`isl_vec_get_element` will return a negative value if anything went wrong. In that case, the value of `*v` is undefined.

The following function can be used to concatenate two vectors.

```

__isl_give isl_vec *isl_vec_concat(__isl_take isl_vec *vec1,
    __isl_take isl_vec *vec2);

```

1.4.19 Matrices

Matrices can be created, copied and freed using the following functions.

```

#include <isl/mat.h>
__isl_give isl_mat *isl_mat_alloc(isl_ctx *ctx,
    unsigned n_row, unsigned n_col);
__isl_give isl_mat *isl_mat_copy(__isl_keep isl_mat *mat);
__isl_null isl_mat *isl_mat_free(__isl_take isl_mat *mat);

```

Note that the elements of a newly created matrix may have arbitrary values. The elements can be changed and inspected using the following functions.

```

isl_ctx *isl_mat_get_ctx(__isl_keep isl_mat *mat);
int isl_mat_rows(__isl_keep isl_mat *mat);
int isl_mat_cols(__isl_keep isl_mat *mat);

```

```

__isl_give isl_val *isl_mat_get_element_val(
    __isl_keep isl_mat *mat, int row, int col);
__isl_give isl_mat *isl_mat_set_element_si(__isl_take isl_mat *mat,
    int row, int col, int v);
__isl_give isl_mat *isl_mat_set_element_val(
    __isl_take isl_mat *mat, int row, int col,
    __isl_take isl_val *v);

```

`isl_mat_get_element` will return a negative value if anything went wrong. In that case, the value of `*v` is undefined.

The following function can be used to compute the (right) inverse of a matrix, i.e., a matrix such that the product of the original and the inverse (in that order) is a multiple of the identity matrix. The input matrix is assumed to be of full row-rank.

```

__isl_give isl_mat *isl_mat_right_inverse(__isl_take isl_mat *mat);

```

The following function can be used to compute the (right) kernel (or null space) of a matrix, i.e., a matrix such that the product of the original and the kernel (in that order) is the zero matrix.

```

__isl_give isl_mat *isl_mat_right_kernel(__isl_take isl_mat *mat);

```

1.4.20 Piecewise Quasi Affine Expressions

The zero quasi affine expression or the quasi affine expression that is equal to a given value or a specified dimension on a given domain can be created using

```

__isl_give isl_aff *isl_aff_zero_on_domain(
    __isl_take isl_local_space *ls);
__isl_give isl_pw_aff *isl_pw_aff_zero_on_domain(
    __isl_take isl_local_space *ls);
__isl_give isl_aff *isl_aff_val_on_domain(
    __isl_take isl_local_space *ls,
    __isl_take isl_val *val);
__isl_give isl_aff *isl_aff_var_on_domain(
    __isl_take isl_local_space *ls,
    enum isl_dim_type type, unsigned pos);
__isl_give isl_pw_aff *isl_pw_aff_var_on_domain(
    __isl_take isl_local_space *ls,
    enum isl_dim_type type, unsigned pos);
__isl_give isl_aff *isl_aff_nan_on_domain(
    __isl_take isl_local_space *ls);
__isl_give isl_pw_aff *isl_pw_aff_nan_on_domain(
    __isl_take isl_local_space *ls);

```

Note that the space in which the resulting objects live is a map space with the given space as domain and a one-dimensional range.

An empty piecewise quasi affine expression (one with no cells) or a piecewise quasi affine expression with a single cell can be created using the following functions.


```

#include <isl/aff.h>
__isl_give isl_pw_aff *isl_pw_aff_empty(
    __isl_take isl_space *space);
__isl_give isl_pw_aff *isl_pw_aff_alloc(
    __isl_take isl_set *set, __isl_take isl_aff *aff);
__isl_give isl_pw_aff *isl_pw_aff_from_aff(
    __isl_take isl_aff *aff);

```

A piecewise quasi affine expression that is equal to 1 on a set and 0 outside the set can be created using the following function.

```

#include <isl/aff.h>
__isl_give isl_pw_aff *isl_set_indicator_function(
    __isl_take isl_set *set);

```

Quasi affine expressions can be copied and freed using

```

#include <isl/aff.h>
__isl_give isl_aff *isl_aff_copy(__isl_keep isl_aff *aff);
__isl_null isl_aff *isl_aff_free(__isl_take isl_aff *aff);

__isl_give isl_pw_aff *isl_pw_aff_copy(
    __isl_keep isl_pw_aff *pwaff);
__isl_null isl_pw_aff *isl_pw_aff_free(
    __isl_take isl_pw_aff *pwaff);

```

A (rational) bound on a dimension can be extracted from an `isl_constraint` using the following function. The constraint is required to have a non-zero coefficient for the specified dimension.

```

#include <isl/constraint.h>
__isl_give isl_aff *isl_constraint_get_bound(
    __isl_keep isl_constraint *constraint,
    enum isl_dim_type type, int pos);

```

The entire affine expression of the constraint can also be extracted using the following function.

```

#include <isl/constraint.h>
__isl_give isl_aff *isl_constraint_get_aff(
    __isl_keep isl_constraint *constraint);

```

Conversely, an equality constraint equating the affine expression to zero or an inequality constraint enforcing the affine expression to be non-negative, can be constructed using

```

__isl_give isl_constraint *isl_equality_from_aff(
    __isl_take isl_aff *aff);
__isl_give isl_constraint *isl_inequality_from_aff(
    __isl_take isl_aff *aff);

```

The expression can be inspected using

```
#include <isl/aff.h>
isl_ctx *isl_aff_get_ctx(__isl_keep isl_aff *aff);
int isl_aff_dim(__isl_keep isl_aff *aff,
               enum isl_dim_type type);
__isl_give isl_local_space *isl_aff_get_domain_local_space(
    __isl_keep isl_aff *aff);
__isl_give isl_local_space *isl_aff_get_local_space(
    __isl_keep isl_aff *aff);
const char *isl_aff_get_dim_name(__isl_keep isl_aff *aff,
                                enum isl_dim_type type, unsigned pos);
const char *isl_pw_aff_get_dim_name(
    __isl_keep isl_pw_aff *pa,
    enum isl_dim_type type, unsigned pos);
int isl_pw_aff_has_dim_id(__isl_keep isl_pw_aff *pa,
                         enum isl_dim_type type, unsigned pos);
__isl_give isl_id *isl_pw_aff_get_dim_id(
    __isl_keep isl_pw_aff *pa,
    enum isl_dim_type type, unsigned pos);
int isl_pw_aff_has_tuple_id(__isl_keep isl_pw_aff *pa,
                           enum isl_dim_type type);
__isl_give isl_id *isl_pw_aff_get_tuple_id(
    __isl_keep isl_pw_aff *pa,
    enum isl_dim_type type);
__isl_give isl_val *isl_aff_get_constant_val(
    __isl_keep isl_aff *aff);
__isl_give isl_val *isl_aff_get_coefficient_val(
    __isl_keep isl_aff *aff,
    enum isl_dim_type type, int pos);
__isl_give isl_val *isl_aff_get_denominator_val(
    __isl_keep isl_aff *aff);
__isl_give isl_aff *isl_aff_get_div(
    __isl_keep isl_aff *aff, int pos);

int isl_pw_aff_n_piece(__isl_keep isl_pw_aff *pwaff);
int isl_pw_aff_foreach_piece(__isl_keep isl_pw_aff *pwaff,
                             int (*fn)(__isl_take isl_set *set,
                                         __isl_take isl_aff *aff,
                                         void *user), void *user);

int isl_aff_is_cst(__isl_keep isl_aff *aff);
int isl_pw_aff_is_cst(__isl_keep isl_pw_aff *pwaff);

int isl_aff_is_nan(__isl_keep isl_aff *aff);
int isl_pw_aff_involves_nan(__isl_keep isl_pw_aff *pa);
```

```

int isl_aff_involves_dims(__isl_keep isl_aff *aff,
    enum isl_dim_type type, unsigned first, unsigned n);
int isl_pw_aff_involves_dims(__isl_keep isl_pw_aff *pwaff,
    enum isl_dim_type type, unsigned first, unsigned n);

isl_ctx *isl_pw_aff_get_ctx(__isl_keep isl_pw_aff *pwaff);
unsigned isl_pw_aff_dim(__isl_keep isl_pw_aff *pwaff,
    enum isl_dim_type type);
int isl_pw_aff_is_empty(__isl_keep isl_pw_aff *pwaff);

```

It can be modified using

```

#include <isl/aff.h>
__isl_give isl_aff *isl_aff_set_tuple_id(
    __isl_take isl_aff *aff,
    enum isl_dim_type type, __isl_take isl_id *id);
__isl_give isl_pw_aff *isl_pw_aff_set_tuple_id(
    __isl_take isl_pw_aff *pwaff,
    enum isl_dim_type type, __isl_take isl_id *id);
__isl_give isl_pw_aff *isl_pw_aff_reset_tuple_id(
    __isl_take isl_pw_aff *pa,
    enum isl_dim_type type);
__isl_give isl_aff *isl_aff_set_dim_name(
    __isl_take isl_aff *aff, enum isl_dim_type type,
    unsigned pos, const char *s);
__isl_give isl_aff *isl_aff_set_dim_id(
    __isl_take isl_aff *aff, enum isl_dim_type type,
    unsigned pos, __isl_take isl_id *id);
__isl_give isl_pw_aff *isl_pw_aff_set_dim_id(
    __isl_take isl_pw_aff *pma,
    enum isl_dim_type type, unsigned pos,
    __isl_take isl_id *id);
__isl_give isl_aff *isl_aff_set_constant_si(
    __isl_take isl_aff *aff, int v);
__isl_give isl_aff *isl_aff_set_constant_val(
    __isl_take isl_aff *aff, __isl_take isl_val *v);
__isl_give isl_aff *isl_aff_set_coefficient_si(
    __isl_take isl_aff *aff,
    enum isl_dim_type type, int pos, int v);
__isl_give isl_aff *isl_aff_set_coefficient_val(
    __isl_take isl_aff *aff,
    enum isl_dim_type type, int pos,
    __isl_take isl_val *v);

__isl_give isl_aff *isl_aff_add_constant_si(
    __isl_take isl_aff *aff, int v);
__isl_give isl_aff *isl_aff_add_constant_val(

```

```

__isl_take isl_aff *aff, __isl_take isl_val *v);
__isl_give isl_aff *isl_aff_add_constant_num_si(
__isl_take isl_aff *aff, int v);
__isl_give isl_aff *isl_aff_add_coefficient_si(
__isl_take isl_aff *aff,
enum isl_dim_type type, int pos, int v);
__isl_give isl_aff *isl_aff_add_coefficient_val(
__isl_take isl_aff *aff,
enum isl_dim_type type, int pos,
__isl_take isl_val *v);

__isl_give isl_aff *isl_aff_insert_dims(
__isl_take isl_aff *aff,
enum isl_dim_type type, unsigned first, unsigned n);
__isl_give isl_pw_aff *isl_pw_aff_insert_dims(
__isl_take isl_pw_aff *pwaff,
enum isl_dim_type type, unsigned first, unsigned n);
__isl_give isl_aff *isl_aff_add_dims(
__isl_take isl_aff *aff,
enum isl_dim_type type, unsigned n);
__isl_give isl_pw_aff *isl_pw_aff_add_dims(
__isl_take isl_pw_aff *pwaff,
enum isl_dim_type type, unsigned n);
__isl_give isl_aff *isl_aff_drop_dims(
__isl_take isl_aff *aff,
enum isl_dim_type type, unsigned first, unsigned n);
__isl_give isl_pw_aff *isl_pw_aff_drop_dims(
__isl_take isl_pw_aff *pwaff,
enum isl_dim_type type, unsigned first, unsigned n);
__isl_give isl_aff *isl_aff_move_dims(
__isl_take isl_aff *aff,
enum isl_dim_type dst_type, unsigned dst_pos,
enum isl_dim_type src_type, unsigned src_pos,
unsigned n);
__isl_give isl_pw_aff *isl_pw_aff_move_dims(
__isl_take isl_pw_aff *pa,
enum isl_dim_type dst_type, unsigned dst_pos,
enum isl_dim_type src_type, unsigned src_pos,
unsigned n);

```

Note that `isl_aff.set_constant_si` and `isl_aff.set_coefficient_si` set the *numerator* of the constant or coefficient, while `isl_aff.set_constant_val` and `isl_aff.set_coefficient_val` set the constant or coefficient as a whole. The `add_constant` and `add_coefficient` functions add an integer or rational value to the possibly rational constant or coefficient. The `add_constant_num` functions add an integer value to the numerator.

To check whether an affine expressions is obviously zero or (obviously) equal to some other affine expression, use

```

#include <isl/aff.h>
int isl_aff_plain_is_zero(__isl_keep isl_aff *aff);
int isl_aff_plain_is_equal(__isl_keep isl_aff *aff1,
    __isl_keep isl_aff *aff2);
int isl_pw_aff_plain_is_equal(
    __isl_keep isl_pw_aff *pwaff1,
    __isl_keep isl_pw_aff *pwaff2);
int isl_pw_aff_is_equal(__isl_keep isl_pw_aff *pa1,
    __isl_keep isl_pw_aff *pa2);
int isl_pw_aff_plain_cmp(__isl_keep isl_pw_aff *pa1,
    __isl_keep isl_pw_aff *pa2);

```

The function `isl_pw_aff_plain_cmp` can be used to sort `isl_pw_affs`. The order is not strictly defined. The current order sorts expressions that only involve earlier dimensions before those that involve later dimensions.

Operations include

```

#include <isl/aff.h>
__isl_give isl_aff *isl_aff_add(__isl_take isl_aff *aff1,
    __isl_take isl_aff *aff2);
__isl_give isl_pw_aff *isl_pw_aff_add(
    __isl_take isl_pw_aff *pwaff1,
    __isl_take isl_pw_aff *pwaff2);
__isl_give isl_pw_aff *isl_pw_aff_min(
    __isl_take isl_pw_aff *pwaff1,
    __isl_take isl_pw_aff *pwaff2);
__isl_give isl_pw_aff *isl_pw_aff_max(
    __isl_take isl_pw_aff *pwaff1,
    __isl_take isl_pw_aff *pwaff2);
__isl_give isl_aff *isl_aff_sub(__isl_take isl_aff *aff1,
    __isl_take isl_aff *aff2);
__isl_give isl_pw_aff *isl_pw_aff_sub(
    __isl_take isl_pw_aff *pwaff1,
    __isl_take isl_pw_aff *pwaff2);
__isl_give isl_aff *isl_aff_neg(__isl_take isl_aff *aff);
__isl_give isl_pw_aff *isl_pw_aff_neg(
    __isl_take isl_pw_aff *pwaff);
__isl_give isl_aff *isl_aff_ceil(__isl_take isl_aff *aff);
__isl_give isl_pw_aff *isl_pw_aff_ceil(
    __isl_take isl_pw_aff *pwaff);
__isl_give isl_aff *isl_aff_floor(__isl_take isl_aff *aff);
__isl_give isl_pw_aff *isl_pw_aff_floor(
    __isl_take isl_pw_aff *pwaff);
__isl_give isl_aff *isl_aff_mod_val(__isl_take isl_aff *aff,
    __isl_take isl_val *mod);
__isl_give isl_pw_aff *isl_pw_aff_mod_val(

```

```

        __isl_take isl_pw_aff *pa,
        __isl_take isl_val *mod);
__isl_give isl_aff *isl_aff_scale_val(__isl_take isl_aff *aff,
        __isl_take isl_val *v);
__isl_give isl_pw_aff *isl_pw_aff_scale_val(
        __isl_take isl_pw_aff *pa, __isl_take isl_val *v);
__isl_give isl_aff *isl_aff_scale_down_ui(
        __isl_take isl_aff *aff, unsigned f);
__isl_give isl_aff *isl_aff_scale_down_val(
        __isl_take isl_aff *aff, __isl_take isl_val *v);
__isl_give isl_pw_aff *isl_pw_aff_scale_down_val(
        __isl_take isl_pw_aff *pa,
        __isl_take isl_val *f);

__isl_give isl_pw_aff *isl_pw_aff_list_min(
        __isl_take isl_pw_aff_list *list);
__isl_give isl_pw_aff *isl_pw_aff_list_max(
        __isl_take isl_pw_aff_list *list);

__isl_give isl_pw_aff *isl_pw_aff_coalesce(
        __isl_take isl_pw_aff *pwqp);

__isl_give isl_aff *isl_aff_align_params(
        __isl_take isl_aff *aff,
        __isl_take isl_space *model);
__isl_give isl_pw_aff *isl_pw_aff_align_params(
        __isl_take isl_pw_aff *pwaff,
        __isl_take isl_space *model);

__isl_give isl_aff *isl_aff_project_domain_on_params(
        __isl_take isl_aff *aff);
__isl_give isl_pw_aff *isl_pw_aff_from_range(
        __isl_take isl_pw_aff *pwa);

__isl_give isl_aff *isl_aff_gist_params(
        __isl_take isl_aff *aff,
        __isl_take isl_set *context);
__isl_give isl_aff *isl_aff_gist(__isl_take isl_aff *aff,
        __isl_take isl_set *context);
__isl_give isl_pw_aff *isl_pw_aff_gist_params(
        __isl_take isl_pw_aff *pwaff,
        __isl_take isl_set *context);
__isl_give isl_pw_aff *isl_pw_aff_gist(
        __isl_take isl_pw_aff *pwaff,
        __isl_take isl_set *context);

__isl_give isl_set *isl_pw_aff_domain(

```

```

        __isl_take isl_pw_aff *pwaff);
__isl_give isl_set *isl_pw_aff_params(
    __isl_take isl_pw_aff *pwa);
__isl_give isl_pw_aff *isl_pw_aff_intersect_domain(
    __isl_take isl_pw_aff *pa,
    __isl_take isl_set *set);
__isl_give isl_pw_aff *isl_pw_aff_intersect_params(
    __isl_take isl_pw_aff *pa,
    __isl_take isl_set *set);

__isl_give isl_aff *isl_aff_mul(__isl_take isl_aff *aff1,
    __isl_take isl_aff *aff2);
__isl_give isl_aff *isl_aff_div(__isl_take isl_aff *aff1,
    __isl_take isl_aff *aff2);
__isl_give isl_pw_aff *isl_pw_aff_mul(
    __isl_take isl_pw_aff *pwaff1,
    __isl_take isl_pw_aff *pwaff2);
__isl_give isl_pw_aff *isl_pw_aff_div(
    __isl_take isl_pw_aff *pa1,
    __isl_take isl_pw_aff *pa2);
__isl_give isl_pw_aff *isl_pw_aff_tdiv_q(
    __isl_take isl_pw_aff *pa1,
    __isl_take isl_pw_aff *pa2);
__isl_give isl_pw_aff *isl_pw_aff_tdiv_r(
    __isl_take isl_pw_aff *pa1,
    __isl_take isl_pw_aff *pa2);

```

When multiplying two affine expressions, at least one of the two needs to be a constant. Similarly, when dividing an affine expression by another, the second expression needs to be a constant. `isl_pw_aff_tdiv_q` computes the quotient of an integer division with rounding towards zero. `isl_pw_aff_tdiv_r` computes the corresponding remainder.

```

#include <isl/aff.h>
__isl_give isl_aff *isl_aff_pullback_aff(
    __isl_take isl_aff *aff1,
    __isl_take isl_aff *aff2);
__isl_give isl_aff *isl_aff_pullback_multi_aff(
    __isl_take isl_aff *aff,
    __isl_take isl_multi_aff *ma);
__isl_give isl_pw_aff *isl_pw_aff_pullback_multi_aff(
    __isl_take isl_pw_aff *pa,
    __isl_take isl_multi_aff *ma);
__isl_give isl_pw_aff *isl_pw_aff_pullback_pw_multi_aff(
    __isl_take isl_pw_aff *pa,
    __isl_take isl_pw_multi_aff *pma);
__isl_give isl_pw_aff *isl_pw_aff_pullback_multi_pw_aff(

```

```

__isl_take isl_pw_aff *pa,
__isl_take isl_multi_pw_aff *mpa);

```

These functions precompose the input expression by the given `isl_aff`, `isl_multi_aff` or `isl_pw_multi_aff`. In other words, the `isl_aff`, `isl_multi_aff` or `isl_pw_multi_aff` is plugged into the (piecewise) affine expression. Objects of type `isl_multi_aff` are described in §1.4.21.

```

#include <isl/aff.h>
__isl_give isl_basic_set *isl_aff_zero_basic_set(
    __isl_take isl_aff *aff);
__isl_give isl_basic_set *isl_aff_neg_basic_set(
    __isl_take isl_aff *aff);
__isl_give isl_basic_set *isl_aff_le_basic_set(
    __isl_take isl_aff *aff1, __isl_take isl_aff *aff2);
__isl_give isl_basic_set *isl_aff_ge_basic_set(
    __isl_take isl_aff *aff1, __isl_take isl_aff *aff2);
__isl_give isl_set *isl_pw_aff_eq_set(
    __isl_take isl_pw_aff *pwaff1,
    __isl_take isl_pw_aff *pwaff2);
__isl_give isl_set *isl_pw_aff_ne_set(
    __isl_take isl_pw_aff *pwaff1,
    __isl_take isl_pw_aff *pwaff2);
__isl_give isl_set *isl_pw_aff_le_set(
    __isl_take isl_pw_aff *pwaff1,
    __isl_take isl_pw_aff *pwaff2);
__isl_give isl_set *isl_pw_aff_lt_set(
    __isl_take isl_pw_aff *pwaff1,
    __isl_take isl_pw_aff *pwaff2);
__isl_give isl_set *isl_pw_aff_ge_set(
    __isl_take isl_pw_aff *pwaff1,
    __isl_take isl_pw_aff *pwaff2);
__isl_give isl_set *isl_pw_aff_gt_set(
    __isl_take isl_pw_aff *pwaff1,
    __isl_take isl_pw_aff *pwaff2);

__isl_give isl_set *isl_pw_aff_list_eq_set(
    __isl_take isl_pw_aff_list *list1,
    __isl_take isl_pw_aff_list *list2);
__isl_give isl_set *isl_pw_aff_list_ne_set(
    __isl_take isl_pw_aff_list *list1,
    __isl_take isl_pw_aff_list *list2);
__isl_give isl_set *isl_pw_aff_list_le_set(
    __isl_take isl_pw_aff_list *list1,
    __isl_take isl_pw_aff_list *list2);
__isl_give isl_set *isl_pw_aff_list_lt_set(
    __isl_take isl_pw_aff_list *list1,

```



```

        __isl_take isl_pw_aff_list *list2);
__isl_give isl_set *isl_pw_aff_list_ge_set(
        __isl_take isl_pw_aff_list *list1,
        __isl_take isl_pw_aff_list *list2);
__isl_give isl_set *isl_pw_aff_list_gt_set(
        __isl_take isl_pw_aff_list *list1,
        __isl_take isl_pw_aff_list *list2);

```

The function `isl_aff_neg_basic_set` returns a basic set containing those elements in the domain space of `aff` where `aff` is negative. The function `isl_aff_ge_basic_set` returns a basic set containing those elements in the shared space of `aff1` and `aff2` where `aff1` is greater than or equal to `aff2`. The function `isl_pw_aff_ge_set` returns a set containing those elements in the shared domain of `pwaff1` and `pwaff2` where `pwaff1` is greater than or equal to `pwaff2`. The functions operating on `isl_pw_aff_list` apply the corresponding `isl_pw_aff` function to each pair of elements in the two lists.

```

#include <isl/aff.h>
__isl_give isl_set *isl_pw_aff_nonneg_set(
        __isl_take isl_pw_aff *pwaff);
__isl_give isl_set *isl_pw_aff_zero_set(
        __isl_take isl_pw_aff *pwaff);
__isl_give isl_set *isl_pw_aff_non_zero_set(
        __isl_take isl_pw_aff *pwaff);

```

The function `isl_pw_aff_nonneg_set` returns a set containing those elements in the domain of `pwaff` where `pwaff` is non-negative.

```

#include <isl/aff.h>
__isl_give isl_pw_aff *isl_pw_aff_cond(
        __isl_take isl_pw_aff *cond,
        __isl_take isl_pw_aff *pwaff_true,
        __isl_take isl_pw_aff *pwaff_false);

```

The function `isl_pw_aff_cond` performs a conditional operator and returns an expression that is equal to `pwaff_true` for elements where `cond` is non-zero and equal to `pwaff_false` for elements where `cond` is zero.

```

#include <isl/aff.h>
__isl_give isl_pw_aff *isl_pw_aff_union_min(
        __isl_take isl_pw_aff *pwaff1,
        __isl_take isl_pw_aff *pwaff2);
__isl_give isl_pw_aff *isl_pw_aff_union_max(
        __isl_take isl_pw_aff *pwaff1,
        __isl_take isl_pw_aff *pwaff2);
__isl_give isl_pw_aff *isl_pw_aff_union_add(
        __isl_take isl_pw_aff *pwaff1,
        __isl_take isl_pw_aff *pwaff2);

```

The function `isl_pw_aff_union_max` computes a piecewise quasi-affine expression with a domain that is the union of those of `pwaff1` and `pwaff2` and such that on each cell, the quasi-affine expression is the maximum of those of `pwaff1` and `pwaff2`. If only one of `pwaff1` or `pwaff2` is defined on a given cell, then the associated expression is the defined one.

An expression can be read from input using

```
#include <isl/aff.h>
__isl_give isl_aff *isl_aff_read_from_str(
    isl_ctx *ctx, const char *str);
__isl_give isl_pw_aff *isl_pw_aff_read_from_str(
    isl_ctx *ctx, const char *str);
```

An expression can be printed using

```
#include <isl/aff.h>
__isl_give isl_printer *isl_printer_print_aff(
    __isl_take isl_printer *p, __isl_keep isl_aff *aff);

__isl_give isl_printer *isl_printer_print_pw_aff(
    __isl_take isl_printer *p,
    __isl_keep isl_pw_aff *pwaff);
```

1.4.21 Piecewise Multiple Quasi Affine Expressions

An `isl_multi_aff` object represents a sequence of zero or more affine expressions, all defined on the same domain space. Similarly, an `isl_multi_pw_aff` object represents a sequence of zero or more piecewise affine expressions.

An `isl_multi_aff` can be constructed from a single `isl_aff` or an `isl_aff_list` using the following functions. Similarly for `isl_multi_pw_aff` and `isl_pw_multi_aff`.

```
#include <isl/aff.h>
__isl_give isl_multi_aff *isl_multi_aff_from_aff(
    __isl_take isl_aff *aff);
__isl_give isl_multi_pw_aff *
isl_multi_pw_aff_from_multi_aff(
    __isl_take isl_multi_aff *ma);
__isl_give isl_multi_pw_aff *isl_multi_pw_aff_from_pw_aff(
    __isl_take isl_pw_aff *pa);
__isl_give isl_pw_multi_aff *isl_pw_multi_aff_from_pw_aff(
    __isl_take isl_pw_aff *pa);
__isl_give isl_multi_aff *isl_multi_aff_from_aff_list(
    __isl_take isl_space *space,
    __isl_take isl_aff_list *list);
```

An `isl_multi_pw_aff` can be converted to an `isl_pw_multi_aff` using the function `isl_pw_multi_aff_from_multi_pw_aff` below. Note however that the domain of the result is the intersection of the domains of the input. The reverse conversion is exact.

```

#include <isl/aff.h>
__isl_give isl_pw_multi_aff *
isl_pw_multi_aff_from_multi_pw_aff(
    __isl_take isl_multi_pw_aff *mpa);
__isl_give isl_multi_pw_aff *
isl_multi_pw_aff_from_pw_multi_aff(
    __isl_take isl_pw_multi_aff *pma);

```

An empty piecewise multiple quasi affine expression (one with no cells), the zero piecewise multiple quasi affine expression (with value zero for each output dimension), a piecewise multiple quasi affine expression with a single cell (with either a universe or a specified domain) or a zero-dimensional piecewise multiple quasi affine expression on a given domain can be created using the following functions.

```

#include <isl/aff.h>
__isl_give isl_pw_multi_aff *isl_pw_multi_aff_empty(
    __isl_take isl_space *space);
__isl_give isl_multi_aff *isl_multi_aff_zero(
    __isl_take isl_space *space);
__isl_give isl_pw_multi_aff *isl_pw_multi_aff_zero(
    __isl_take isl_space *space);
__isl_give isl_multi_pw_aff *isl_multi_pw_aff_zero(
    __isl_take isl_space *space);
__isl_give isl_multi_aff *isl_multi_aff_identity(
    __isl_take isl_space *space);
__isl_give isl_pw_multi_aff *isl_pw_multi_aff_identity(
    __isl_take isl_space *space);
__isl_give isl_multi_pw_aff *isl_multi_pw_aff_identity(
    __isl_take isl_space *space);
__isl_give isl_multi_aff *isl_multi_aff_domain_map(
    __isl_take isl_space *space);
__isl_give isl_multi_aff *isl_multi_aff_range_map(
    __isl_take isl_space *space);
__isl_give isl_multi_aff *isl_multi_aff_project_out_map(
    __isl_take isl_space *space,
    enum isl_dim_type type,
    unsigned first, unsigned n);
__isl_give isl_pw_multi_aff *
isl_pw_multi_aff_project_out_map(
    __isl_take isl_space *space,
    enum isl_dim_type type,
    unsigned first, unsigned n);
__isl_give isl_pw_multi_aff *
isl_pw_multi_aff_from_multi_aff(
    __isl_take isl_multi_aff *ma);
__isl_give isl_pw_multi_aff *isl_pw_multi_aff_alloc(

```

```

        __isl_take isl_set *set,
        __isl_take isl_multi_aff *maff);
__isl_give isl_pw_multi_aff *isl_pw_multi_aff_from_domain(
    __isl_take isl_set *set);

__isl_give isl_union_pw_multi_aff *
isl_union_pw_multi_aff_empty(
    __isl_take isl_space *space);
__isl_give isl_union_pw_multi_aff *
isl_union_pw_multi_aff_add_pw_multi_aff(
    __isl_take isl_union_pw_multi_aff *upma,
    __isl_take isl_pw_multi_aff *pma);
__isl_give isl_union_pw_multi_aff *
isl_union_pw_multi_aff_from_domain(
    __isl_take isl_union_set *uset);

```

A piecewise multiple quasi affine expression can also be initialized from an `isl_set` or `isl_map`, provided the `isl_set` is a singleton and the `isl_map` is single-valued. In case of a conversion from an `isl_union_set` or an `isl_union_map` to an `isl_union_pw_multi_aff`, these properties need to hold in each space.

```

__isl_give isl_pw_multi_aff *isl_pw_multi_aff_from_set(
    __isl_take isl_set *set);
__isl_give isl_pw_multi_aff *isl_pw_multi_aff_from_map(
    __isl_take isl_map *map);

__isl_give isl_union_pw_multi_aff *
isl_union_pw_multi_aff_from_union_set(
    __isl_take isl_union_set *uset);
__isl_give isl_union_pw_multi_aff *
isl_union_pw_multi_aff_from_union_map(
    __isl_take isl_union_map *umap);

```

Multiple quasi affine expressions can be copied and freed using

```

#include <isl/aff.h>
__isl_give isl_multi_aff *isl_multi_aff_copy(
    __isl_keep isl_multi_aff *maff);
__isl_null isl_multi_aff *isl_multi_aff_free(
    __isl_take isl_multi_aff *maff);

__isl_give isl_pw_multi_aff *isl_pw_multi_aff_copy(
    __isl_keep isl_pw_multi_aff *pma);
__isl_null isl_pw_multi_aff *isl_pw_multi_aff_free(
    __isl_take isl_pw_multi_aff *pma);

__isl_give isl_union_pw_multi_aff *

```

```

isl_union_pw_multi_aff_copy(
    __isl_keep isl_union_pw_multi_aff *upma);
__isl_null isl_union_pw_multi_aff *
isl_union_pw_multi_aff_free(
    __isl_take isl_union_pw_multi_aff *upma);

__isl_give isl_multi_pw_aff *isl_multi_pw_aff_copy(
    __isl_keep isl_multi_pw_aff *mpa);
__isl_null isl_multi_pw_aff *isl_multi_pw_aff_free(
    __isl_take isl_multi_pw_aff *mpa);

```

The expression can be inspected using

```

#include <isl/aff.h>
isl_ctx *isl_multi_aff_get_ctx(
    __isl_keep isl_multi_aff *maff);
isl_ctx *isl_pw_multi_aff_get_ctx(
    __isl_keep isl_pw_multi_aff *pma);
isl_ctx *isl_union_pw_multi_aff_get_ctx(
    __isl_keep isl_union_pw_multi_aff *upma);
isl_ctx *isl_multi_pw_aff_get_ctx(
    __isl_keep isl_multi_pw_aff *mpa);

int isl_multi_aff_involves_dims(
    __isl_keep isl_multi_aff *ma,
    enum isl_dim_type type, unsigned first, unsigned n);
int isl_multi_pw_aff_involves_dims(
    __isl_keep isl_multi_pw_aff *mpa,
    enum isl_dim_type type, unsigned first, unsigned n);

unsigned isl_multi_aff_dim(__isl_keep isl_multi_aff *maff,
    enum isl_dim_type type);
unsigned isl_pw_multi_aff_dim(
    __isl_keep isl_pw_multi_aff *pma,
    enum isl_dim_type type);
unsigned isl_multi_pw_aff_dim(
    __isl_keep isl_multi_pw_aff *mpa,
    enum isl_dim_type type);
__isl_give isl_aff *isl_multi_aff_get_aff(
    __isl_keep isl_multi_aff *multi, int pos);
__isl_give isl_pw_aff *isl_pw_multi_aff_get_pw_aff(
    __isl_keep isl_pw_multi_aff *pma, int pos);
__isl_give isl_pw_aff *isl_multi_pw_aff_get_pw_aff(
    __isl_keep isl_multi_pw_aff *mpa, int pos);
int isl_multi_aff_find_dim_by_id(
    __isl_keep isl_multi_aff *ma,
    enum isl_dim_type type, __isl_keep isl_id *id);

```

```

int isl_multi_pw_aff_find_dim_by_id(
    __isl_keep isl_multi_pw_aff *mpa,
    enum isl_dim_type type, __isl_keep isl_id *id);
const char *isl_pw_multi_aff_get_dim_name(
    __isl_keep isl_pw_multi_aff *pma,
    enum isl_dim_type type, unsigned pos);
__isl_give isl_id *isl_multi_aff_get_dim_id(
    __isl_keep isl_multi_aff *ma,
    enum isl_dim_type type, unsigned pos);
__isl_give isl_id *isl_pw_multi_aff_get_dim_id(
    __isl_keep isl_pw_multi_aff *pma,
    enum isl_dim_type type, unsigned pos);
__isl_give isl_id *isl_multi_pw_aff_get_dim_id(
    __isl_keep isl_multi_pw_aff *mpa,
    enum isl_dim_type type, unsigned pos);
const char *isl_multi_aff_get_tuple_name(
    __isl_keep isl_multi_aff *multi,
    enum isl_dim_type type);
int isl_pw_multi_aff_has_tuple_name(
    __isl_keep isl_pw_multi_aff *pma,
    enum isl_dim_type type);
const char *isl_pw_multi_aff_get_tuple_name(
    __isl_keep isl_pw_multi_aff *pma,
    enum isl_dim_type type);
int isl_multi_aff_has_tuple_id(__isl_keep isl_multi_aff *ma,
    enum isl_dim_type type);
int isl_pw_multi_aff_has_tuple_id(
    __isl_keep isl_pw_multi_aff *pma,
    enum isl_dim_type type);
int isl_multi_pw_aff_has_tuple_id(
    __isl_keep isl_multi_pw_aff *mpa,
    enum isl_dim_type type);
__isl_give isl_id *isl_multi_aff_get_tuple_id(
    __isl_keep isl_multi_aff *ma,
    enum isl_dim_type type);
__isl_give isl_id *isl_pw_multi_aff_get_tuple_id(
    __isl_keep isl_pw_multi_aff *pma,
    enum isl_dim_type type);
__isl_give isl_id *isl_multi_pw_aff_get_tuple_id(
    __isl_keep isl_multi_pw_aff *mpa,
    enum isl_dim_type type);
int isl_multi_aff_range_is_wrapping(
    __isl_keep isl_multi_aff *ma);
int isl_multi_pw_aff_range_is_wrapping(
    __isl_keep isl_multi_pw_aff *mpa);

```

```

int isl_pw_multi_aff_foreach_piece(
    __isl_keep isl_pw_multi_aff *pma,
    int (*fn)(__isl_take isl_set *set,
               __isl_take isl_multi_aff *maff,
               void *user), void *user);

int isl_union_pw_multi_aff_foreach_pw_multi_aff(
    __isl_keep isl_union_pw_multi_aff *upma,
    int (*fn)(__isl_take isl_pw_multi_aff *pma,
               void *user), void *user);

```

It can be modified using

```

#include <isl/aff.h>
__isl_give isl_multi_aff *isl_multi_aff_set_aff(
    __isl_take isl_multi_aff *multi, int pos,
    __isl_take isl_aff *aff);
__isl_give isl_pw_multi_aff *isl_pw_multi_aff_set_pw_aff(
    __isl_take isl_pw_multi_aff *pma, unsigned pos,
    __isl_take isl_pw_aff *pa);
__isl_give isl_multi_aff *isl_multi_aff_set_dim_name(
    __isl_take isl_multi_aff *maff,
    enum isl_dim_type type, unsigned pos, const char *s);
__isl_give isl_multi_aff *isl_multi_aff_set_dim_id(
    __isl_take isl_multi_aff *maff,
    enum isl_dim_type type, unsigned pos,
    __isl_take isl_id *id);
__isl_give isl_multi_aff *isl_multi_aff_set_tuple_name(
    __isl_take isl_multi_aff *maff,
    enum isl_dim_type type, const char *s);
__isl_give isl_multi_aff *isl_multi_aff_set_tuple_id(
    __isl_take isl_multi_aff *maff,
    enum isl_dim_type type, __isl_take isl_id *id);
__isl_give isl_pw_multi_aff *isl_pw_multi_aff_set_tuple_id(
    __isl_take isl_pw_multi_aff *pma,
    enum isl_dim_type type, __isl_take isl_id *id);
__isl_give isl_multi_aff *isl_multi_aff_reset_tuple_id(
    __isl_take isl_multi_aff *ma,
    enum isl_dim_type type);
__isl_give isl_multi_pw_aff *
isl_multi_pw_aff_reset_tuple_id(
    __isl_take isl_multi_pw_aff *mpa,
    enum isl_dim_type type);
__isl_give isl_multi_aff *isl_multi_aff_reset_user(
    __isl_take isl_multi_aff *ma);
__isl_give isl_multi_pw_aff *isl_multi_pw_aff_reset_user(
    __isl_take isl_multi_pw_aff *mpa);

```

```

__isl_give isl_multi_pw_aff *
isl_multi_pw_aff_set_dim_name(
    __isl_take isl_multi_pw_aff *mpa,
    enum isl_dim_type type, unsigned pos, const char *s);
__isl_give isl_multi_pw_aff *
isl_multi_pw_aff_set_dim_id(
    __isl_take isl_multi_pw_aff *mpa,
    enum isl_dim_type type, unsigned pos,
    __isl_take isl_id *id);
__isl_give isl_multi_pw_aff *
isl_multi_pw_aff_set_tuple_name(
    __isl_take isl_multi_pw_aff *mpa,
    enum isl_dim_type type, const char *s);

__isl_give isl_multi_aff *isl_multi_aff_flatten_domain(
    __isl_take isl_multi_aff *ma);

__isl_give isl_multi_aff *isl_multi_aff_insert_dims(
    __isl_take isl_multi_aff *ma,
    enum isl_dim_type type, unsigned first, unsigned n);
__isl_give isl_multi_aff *isl_multi_aff_add_dims(
    __isl_take isl_multi_aff *ma,
    enum isl_dim_type type, unsigned n);
__isl_give isl_multi_aff *isl_multi_aff_drop_dims(
    __isl_take isl_multi_aff *maff,
    enum isl_dim_type type, unsigned first, unsigned n);
__isl_give isl_pw_multi_aff *isl_pw_multi_aff_drop_dims(
    __isl_take isl_pw_multi_aff *pma,
    enum isl_dim_type type, unsigned first, unsigned n);

__isl_give isl_multi_pw_aff *isl_multi_pw_aff_insert_dims(
    __isl_take isl_multi_pw_aff *mpa,
    enum isl_dim_type type, unsigned first, unsigned n);
__isl_give isl_multi_pw_aff *isl_multi_pw_aff_add_dims(
    __isl_take isl_multi_pw_aff *mpa,
    enum isl_dim_type type, unsigned n);
__isl_give isl_multi_pw_aff *isl_multi_pw_aff_move_dims(
    __isl_take isl_multi_pw_aff *pma,
    enum isl_dim_type dst_type, unsigned dst_pos,
    enum isl_dim_type src_type, unsigned src_pos,
    unsigned n);

```

To check whether two multiple affine expressions are (obviously) equal to each other, use

```

int isl_multi_aff_plain_is_equal(__isl_keep isl_multi_aff *maff1,
    __isl_keep isl_multi_aff *maff2);

```



```

int isl_pw_multi_aff_plain_is_equal(
    __isl_keep isl_pw_multi_aff *pma1,
    __isl_keep isl_pw_multi_aff *pma2);
int isl_multi_pw_aff_plain_is_equal(
    __isl_keep isl_multi_pw_aff *mpa1,
    __isl_keep isl_multi_pw_aff *mpa2);
int isl_multi_pw_aff_is_equal(
    __isl_keep isl_multi_pw_aff *mpa1,
    __isl_keep isl_multi_pw_aff *mpa2);

```

Operations include

```

#include <isl/aff.h>
__isl_give isl_pw_multi_aff *isl_pw_multi_aff_union_lexmin(
    __isl_take isl_pw_multi_aff *pma1,
    __isl_take isl_pw_multi_aff *pma2);
__isl_give isl_pw_multi_aff *isl_pw_multi_aff_union_lexmax(
    __isl_take isl_pw_multi_aff *pma1,
    __isl_take isl_pw_multi_aff *pma2);
__isl_give isl_multi_aff *isl_multi_aff_floor(
    __isl_take isl_multi_aff *ma);
__isl_give isl_multi_aff *isl_multi_aff_add(
    __isl_take isl_multi_aff *maff1,
    __isl_take isl_multi_aff *maff2);
__isl_give isl_pw_multi_aff *isl_pw_multi_aff_add(
    __isl_take isl_pw_multi_aff *pma1,
    __isl_take isl_pw_multi_aff *pma2);
__isl_give isl_union_pw_multi_aff *isl_union_pw_multi_aff_add(
    __isl_take isl_union_pw_multi_aff *upma1,
    __isl_take isl_union_pw_multi_aff *upma2);
__isl_give isl_pw_multi_aff *isl_pw_multi_aff_union_add(
    __isl_take isl_pw_multi_aff *pma1,
    __isl_take isl_pw_multi_aff *pma2);
__isl_give isl_multi_aff *isl_multi_aff_sub(
    __isl_take isl_multi_aff *ma1,
    __isl_take isl_multi_aff *ma2);
__isl_give isl_pw_multi_aff *isl_pw_multi_aff_sub(
    __isl_take isl_pw_multi_aff *pma1,
    __isl_take isl_pw_multi_aff *pma2);
__isl_give isl_union_pw_multi_aff *isl_union_pw_multi_aff_sub(
    __isl_take isl_union_pw_multi_aff *upma1,
    __isl_take isl_union_pw_multi_aff *upma2);

```

`isl_multi_aff_sub` subtracts the second argument from the first.

```

__isl_give isl_multi_aff *isl_multi_aff_scale_val(
    __isl_take isl_multi_aff *ma,

```

```

        __isl_take isl_val *v);
__isl_give isl_pw_multi_aff *isl_pw_multi_aff_scale_val(
    __isl_take isl_pw_multi_aff *pma,
    __isl_take isl_val *v);
__isl_give isl_multi_pw_aff *isl_multi_pw_aff_scale_val(
    __isl_take isl_multi_pw_aff *mpa,
    __isl_take isl_val *v);
__isl_give isl_multi_aff *isl_multi_aff_scale_multi_val(
    __isl_take isl_multi_aff *ma,
    __isl_take isl_multi_val *mv);
__isl_give isl_pw_multi_aff *
isl_pw_multi_aff_scale_multi_val(
    __isl_take isl_pw_multi_aff *pma,
    __isl_take isl_multi_val *mv);
__isl_give isl_multi_pw_aff *
isl_multi_pw_aff_scale_multi_val(
    __isl_take isl_multi_pw_aff *mpa,
    __isl_take isl_multi_val *mv);
__isl_give isl_union_pw_multi_aff *
isl_union_pw_multi_aff_scale_multi_val(
    __isl_take isl_union_pw_multi_aff *upma,
    __isl_take isl_multi_val *mv);
__isl_give isl_multi_aff *
isl_multi_aff_scale_down_multi_val(
    __isl_take isl_multi_aff *ma,
    __isl_take isl_multi_val *mv);
__isl_give isl_multi_pw_aff *
isl_multi_pw_aff_scale_down_multi_val(
    __isl_take isl_multi_pw_aff *mpa,
    __isl_take isl_multi_val *mv);

```

`isl_multi_aff_scale_multi_val` scales the elements of `ma` by the corresponding elements of `mv`.

```

__isl_give isl_pw_multi_aff *isl_pw_multi_aff_fix_si(
    __isl_take isl_pw_multi_aff *pma,
    enum isl_dim_type type, unsigned pos, int value);
__isl_give isl_pw_multi_aff *isl_pw_multi_aff_intersect_params(
    __isl_take isl_pw_multi_aff *pma,
    __isl_take isl_set *set);
__isl_give isl_set *isl_multi_pw_aff_domain(
    __isl_take isl_multi_pw_aff *mpa);
__isl_give isl_multi_pw_aff *
isl_multi_pw_aff_intersect_params(
    __isl_take isl_multi_pw_aff *mpa,
    __isl_take isl_set *set);

```

```

__isl_give isl_pw_multi_aff *isl_pw_multi_aff_intersect_domain(
    __isl_take isl_pw_multi_aff *pma,
    __isl_take isl_set *set);
__isl_give isl_multi_pw_aff *
isl_multi_pw_aff_intersect_domain(
    __isl_take isl_multi_pw_aff *mpa,
    __isl_take isl_set *domain);
__isl_give isl_union_pw_multi_aff *
isl_union_pw_multi_aff_intersect_domain(
    __isl_take isl_union_pw_multi_aff *upma,
    __isl_take isl_union_set *uset);
__isl_give isl_multi_aff *isl_multi_aff_lift(
    __isl_take isl_multi_aff *maff,
    __isl_give isl_local_space **ls);
__isl_give isl_pw_multi_aff *isl_pw_multi_aff_coalesce(
    __isl_take isl_pw_multi_aff *pma);
__isl_give isl_multi_pw_aff *isl_multi_pw_aff_coalesce(
    __isl_take isl_multi_pw_aff *mpa);
__isl_give isl_multi_aff *isl_multi_aff_align_params(
    __isl_take isl_multi_aff *multi,
    __isl_take isl_space *model);
__isl_give isl_pw_multi_aff *isl_pw_multi_aff_align_params(
    __isl_take isl_pw_multi_aff *pma,
    __isl_take isl_space *model);
__isl_give isl_union_pw_multi_aff *
isl_union_pw_multi_aff_align_params(
    __isl_take isl_union_pw_multi_aff *upma,
    __isl_take isl_space *model);
__isl_give isl_pw_multi_aff *
isl_pw_multi_aff_project_domain_on_params(
    __isl_take isl_pw_multi_aff *pma);
__isl_give isl_multi_aff *isl_multi_aff_gist_params(
    __isl_take isl_multi_aff *maff,
    __isl_take isl_set *context);
__isl_give isl_multi_aff *isl_multi_aff_gist(
    __isl_take isl_multi_aff *maff,
    __isl_take isl_set *context);
__isl_give isl_pw_multi_aff *isl_pw_multi_aff_gist_params(
    __isl_take isl_pw_multi_aff *pma,
    __isl_take isl_set *set);
__isl_give isl_pw_multi_aff *isl_pw_multi_aff_gist(
    __isl_take isl_pw_multi_aff *pma,
    __isl_take isl_set *set);
__isl_give isl_multi_pw_aff *isl_multi_pw_aff_gist_params(
    __isl_take isl_multi_pw_aff *mpa,
    __isl_take isl_set *set);

```

```

__isl_give isl_multi_pw_aff *isl_multi_pw_aff_gist(
    __isl_take isl_multi_pw_aff *mpa,
    __isl_take isl_set *set);
__isl_give isl_multi_aff *isl_multi_aff_from_range(
    __isl_take isl_multi_aff *ma);
__isl_give isl_multi_pw_aff *isl_multi_pw_aff_from_range(
    __isl_take isl_multi_pw_aff *mpa);
__isl_give isl_set *isl_pw_multi_aff_domain(
    __isl_take isl_pw_multi_aff *pma);
__isl_give isl_union_set *isl_union_pw_multi_aff_domain(
    __isl_take isl_union_pw_multi_aff *upma);
__isl_give isl_multi_aff *isl_multi_aff_range_splice(
    __isl_take isl_multi_aff *ma1, unsigned pos,
    __isl_take isl_multi_aff *ma2);
__isl_give isl_multi_aff *isl_multi_aff_splice(
    __isl_take isl_multi_aff *ma1,
    unsigned in_pos, unsigned out_pos,
    __isl_take isl_multi_aff *ma2);
__isl_give isl_multi_aff *isl_multi_aff_range_product(
    __isl_take isl_multi_aff *ma1,
    __isl_take isl_multi_aff *ma2);
__isl_give isl_multi_aff *
isl_multi_aff_range_factor_domain(
    __isl_take isl_multi_aff *ma);
__isl_give isl_multi_aff *
isl_multi_aff_range_factor_range(
    __isl_take isl_multi_aff *ma);
__isl_give isl_multi_aff *isl_multi_aff_flat_range_product(
    __isl_take isl_multi_aff *ma1,
    __isl_take isl_multi_aff *ma2);
__isl_give isl_multi_aff *isl_multi_aff_product(
    __isl_take isl_multi_aff *ma1,
    __isl_take isl_multi_aff *ma2);
__isl_give isl_multi_pw_aff *isl_multi_pw_aff_product(
    __isl_take isl_multi_pw_aff *mpa1,
    __isl_take isl_multi_pw_aff *mpa2);
__isl_give isl_pw_multi_aff *
isl_pw_multi_aff_range_product(
    __isl_take isl_pw_multi_aff *pma1,
    __isl_take isl_pw_multi_aff *pma2);
__isl_give isl_multi_pw_aff *
isl_multi_pw_aff_range_factor_domain(
    __isl_take isl_multi_pw_aff *mpa);
__isl_give isl_multi_pw_aff *
isl_multi_pw_aff_range_factor_range(
    __isl_take isl_multi_pw_aff *mpa);

```

```

__isl_give isl_pw_multi_aff *
isl_pw_multi_aff_flat_range_product(
    __isl_take isl_pw_multi_aff *pma1,
    __isl_take isl_pw_multi_aff *pma2);
__isl_give isl_pw_multi_aff *isl_pw_multi_aff_product(
    __isl_take isl_pw_multi_aff *pma1,
    __isl_take isl_pw_multi_aff *pma2);
__isl_give isl_union_pw_multi_aff *
isl_union_pw_multi_aff_flat_range_product(
    __isl_take isl_union_pw_multi_aff *upma1,
    __isl_take isl_union_pw_multi_aff *upma2);
__isl_give isl_multi_pw_aff *
isl_multi_pw_aff_range_splice(
    __isl_take isl_multi_pw_aff *mpa1, unsigned pos,
    __isl_take isl_multi_pw_aff *mpa2);
__isl_give isl_multi_pw_aff *isl_multi_pw_aff_splice(
    __isl_take isl_multi_pw_aff *mpa1,
    unsigned in_pos, unsigned out_pos,
    __isl_take isl_multi_pw_aff *mpa2);
__isl_give isl_multi_pw_aff *
isl_multi_pw_aff_range_product(
    __isl_take isl_multi_pw_aff *mpa1,
    __isl_take isl_multi_pw_aff *mpa2);
__isl_give isl_multi_pw_aff *
isl_multi_pw_aff_flat_range_product(
    __isl_take isl_multi_pw_aff *mpa1,
    __isl_take isl_multi_pw_aff *mpa2);

```

If the `ls` argument of `isl_multi_aff_lift` is not `NULL`, then it is assigned the local space that lies at the basis of the lifting applied.

```

#include <isl/aff.h>
__isl_give isl_multi_aff *isl_multi_aff_pullback_multi_aff(
    __isl_take isl_multi_aff *ma1,
    __isl_take isl_multi_aff *ma2);
__isl_give isl_pw_multi_aff *
isl_pw_multi_aff_pullback_multi_aff(
    __isl_take isl_pw_multi_aff *pma,
    __isl_take isl_multi_aff *ma);
__isl_give isl_multi_pw_aff *
isl_multi_pw_aff_pullback_multi_aff(
    __isl_take isl_multi_pw_aff *mpa,
    __isl_take isl_multi_aff *ma);
__isl_give isl_pw_multi_aff *
isl_pw_multi_aff_pullback_pw_multi_aff(
    __isl_take isl_pw_multi_aff *pma1,

```

```

        __isl_take isl_pw_multi_aff *pma2);
__isl_give isl_multi_pw_aff *
isl_multi_pw_aff_pullback_pw_multi_aff(
    __isl_take isl_multi_pw_aff *mpa,
    __isl_take isl_pw_multi_aff *pma);
__isl_give isl_multi_pw_aff *
isl_multi_pw_aff_pullback_multi_pw_aff(
    __isl_take isl_multi_pw_aff *mpa1,
    __isl_take isl_multi_pw_aff *mpa2);

```

The function `isl_multi_aff_pullback_multi_aff` precomposes `ma1` by `ma2`. In other words, `ma2` is plugged into `ma1`.

```

__isl_give isl_set *isl_multi_aff_lex_le_set(
    __isl_take isl_multi_aff *ma1,
    __isl_take isl_multi_aff *ma2);
__isl_give isl_set *isl_multi_aff_lex_ge_set(
    __isl_take isl_multi_aff *ma1,
    __isl_take isl_multi_aff *ma2);

```

The function `isl_multi_aff_lex_le_set` returns a set containing those elements in the shared domain space where `ma1` is lexicographically smaller than or equal to `ma2`.

An expression can be read from input using

```

#include <isl/aff.h>
__isl_give isl_multi_aff *isl_multi_aff_read_from_str(
    isl_ctx *ctx, const char *str);
__isl_give isl_pw_multi_aff *isl_pw_multi_aff_read_from_str(
    isl_ctx *ctx, const char *str);
__isl_give isl_multi_pw_aff *isl_multi_pw_aff_read_from_str(
    isl_ctx *ctx, const char *str);
__isl_give isl_union_pw_multi_aff *
isl_union_pw_multi_aff_read_from_str(
    isl_ctx *ctx, const char *str);

```

An expression can be printed using

```

#include <isl/aff.h>
__isl_give isl_printer *isl_printer_print_multi_aff(
    __isl_take isl_printer *p,
    __isl_keep isl_multi_aff *maff);
__isl_give isl_printer *isl_printer_print_pw_multi_aff(
    __isl_take isl_printer *p,
    __isl_keep isl_pw_multi_aff *pma);
__isl_give isl_printer *isl_printer_print_union_pw_multi_aff(
    __isl_take isl_printer *p,

```

```

__isl_keep isl_union_pw_multi_aff *upma);
__isl_give isl_printer *isl_printer_print_multi_pw_aff(
__isl_take isl_printer *p,
__isl_keep isl_multi_pw_aff *mpa);

```

1.4.22 Points

Points are elements of a set. They can be used to construct simple sets (boxes) or they can be used to represent the individual elements of a set. The zero point (the origin) can be created using

```

__isl_give isl_point *isl_point_zero(__isl_take isl_space *space);

```

The coordinates of a point can be inspected, set and changed using

```

__isl_give isl_val *isl_point_get_coordinate_val(
__isl_keep isl_point *pnt,
enum isl_dim_type type, int pos);
__isl_give isl_point *isl_point_set_coordinate_val(
__isl_take isl_point *pnt,
enum isl_dim_type type, int pos,
__isl_take isl_val *v);

__isl_give isl_point *isl_point_add_ui(
__isl_take isl_point *pnt,
enum isl_dim_type type, int pos, unsigned val);
__isl_give isl_point *isl_point_sub_ui(
__isl_take isl_point *pnt,
enum isl_dim_type type, int pos, unsigned val);

```

Other properties can be obtained using

```

isl_ctx *isl_point_get_ctx(__isl_keep isl_point *pnt);

```

Points can be copied or freed using

```

__isl_give isl_point *isl_point_copy(
__isl_keep isl_point *pnt);
void isl_point_free(__isl_take isl_point *pnt);

```

A singleton set can be created from a point using

```

__isl_give isl_basic_set *isl_basic_set_from_point(
__isl_take isl_point *pnt);
__isl_give isl_set *isl_set_from_point(
__isl_take isl_point *pnt);

```

and a box can be created from two opposite extremal points using

```

__isl_give isl_basic_set *isl_basic_set_box_from_points(
    __isl_take isl_point *pnt1,
    __isl_take isl_point *pnt2);
__isl_give isl_set *isl_set_box_from_points(
    __isl_take isl_point *pnt1,
    __isl_take isl_point *pnt2);

```

All elements of a **bounded** (union) set can be enumerated using the following functions.

```

int isl_set_foreach_point(__isl_keep isl_set *set,
    int (*fn)(__isl_take isl_point *pnt, void *user),
    void *user);
int isl_union_set_foreach_point(__isl_keep isl_union_set *uset,
    int (*fn)(__isl_take isl_point *pnt, void *user),
    void *user);

```

The function `fn` is called for each integer point in `set` with as second argument the last argument of the `isl_set_foreach_point` call. The function `fn` should return 0 on success and -1 on failure. In the latter case, `isl_set_foreach_point` will stop enumerating and return -1 as well. If the enumeration is performed successfully and to completion, then `isl_set_foreach_point` returns 0.

To obtain a single point of a (basic) set, use

```

__isl_give isl_point *isl_basic_set_sample_point(
    __isl_take isl_basic_set *bset);
__isl_give isl_point *isl_set_sample_point(
    __isl_take isl_set *set);

```

If `set` does not contain any (integer) points, then the resulting point will be “void”, a property that can be tested using

```

int isl_point_is_void(__isl_keep isl_point *pnt);

```

1.4.23 Piecewise Quasipolynomials

A piecewise quasipolynomial is a particular kind of function that maps a parametric point to a rational value. More specifically, a quasipolynomial is a polynomial expression in greatest integer parts of affine expressions of parameters and variables. A piecewise quasipolynomial is a subdivision of a given parametric domain into disjoint cells with a quasipolynomial associated to each cell. The value of the piecewise quasipolynomial at a given point is the value of the quasipolynomial associated to the cell that contains the point. Outside of the union of cells, the value is assumed to be zero. For example, the piecewise quasipolynomial

$$[n] \rightarrow \{ [x] \rightarrow ((1 + n) - x) : x \leq n \text{ and } x \geq 0 \}$$

maps x to $1 + n - x$ for values of x between 0 and n . A given piecewise quasipolynomial has a fixed domain dimension. Union piecewise quasipolynomials are used to contain piecewise quasipolynomials defined over different domains. Piecewise quasipolynomials are mainly used by the `barvinok` library for representing the number of elements in a parametric set or map. For example, the piecewise quasipolynomial above represents the number of points in the map

$$[n] \rightarrow \{ [x] \rightarrow [y] : x, y \geq 0 \text{ and } 0 \leq x + y \leq n \}$$

Input and Output

Piecewise quasipolynomials can be read from input using

```
__isl_give isl_union_pw_qpolynomial *
isl_union_pw_qpolynomial_read_from_str(
    isl_ctx *ctx, const char *str);
```

Quasipolynomials and piecewise quasipolynomials can be printed using the following functions.

```
__isl_give isl_printer *isl_printer_print_qpolynomial(
    __isl_take isl_printer *p,
    __isl_keep isl_qpolynomial *qp);

__isl_give isl_printer *isl_printer_print_pw_qpolynomial(
    __isl_take isl_printer *p,
    __isl_keep isl_pw_qpolynomial *pwqp);

__isl_give isl_printer *isl_printer_print_union_pw_qpolynomial(
    __isl_take isl_printer *p,
    __isl_keep isl_union_pw_qpolynomial *upwqp);
```

The output format of the printer needs to be set to either `ISL_FORMAT_ISL` or `ISL_FORMAT_C`. For `isl_printer_print_union_pw_qpolynomial`, only `ISL_FORMAT_ISL` is supported. In case of printing in `ISL_FORMAT_C`, the user may want to set the names of all dimensions

```
__isl_give isl_qpolynomial *isl_qpolynomial_set_dim_name(
    __isl_take isl_qpolynomial *qp,
    enum isl_dim_type type, unsigned pos,
    const char *s);
__isl_give isl_pw_qpolynomial *
isl_pw_qpolynomial_set_dim_name(
    __isl_take isl_pw_qpolynomial *pwqp,
    enum isl_dim_type type, unsigned pos,
    const char *s);
```

Creating New (Piecewise) Quasipolynomials

Some simple quasipolynomials can be created using the following functions. More complicated quasipolynomials can be created by applying operations such as addition and multiplication on the resulting quasipolynomials

```
__isl_give isl_qpolynomial *isl_qpolynomial_zero_on_domain(
    __isl_take isl_space *domain);
__isl_give isl_qpolynomial *isl_qpolynomial_one_on_domain(
    __isl_take isl_space *domain);
__isl_give isl_qpolynomial *isl_qpolynomial_infty_on_domain(
    __isl_take isl_space *domain);
__isl_give isl_qpolynomial *isl_qpolynomial_neginfty_on_domain(
    __isl_take isl_space *domain);
__isl_give isl_qpolynomial *isl_qpolynomial_nan_on_domain(
    __isl_take isl_space *domain);
__isl_give isl_qpolynomial *isl_qpolynomial_val_on_domain(
    __isl_take isl_space *domain,
    __isl_take isl_val *val);
__isl_give isl_qpolynomial *isl_qpolynomial_var_on_domain(
    __isl_take isl_space *domain,
    enum isl_dim_type type, unsigned pos);
__isl_give isl_qpolynomial *isl_qpolynomial_from_aff(
    __isl_take isl_aff *aff);
```

Note that the space in which a quasipolynomial lives is a map space with a one-dimensional range. The domain argument in some of the functions above corresponds to the domain of this map space.

The zero piecewise quasipolynomial or a piecewise quasipolynomial with a single cell can be created using the following functions. Multiple of these single cell piecewise quasipolynomials can be combined to create more complicated piecewise quasipolynomials.

```
__isl_give isl_pw_qpolynomial *isl_pw_qpolynomial_zero(
    __isl_take isl_space *space);
__isl_give isl_pw_qpolynomial *isl_pw_qpolynomial_alloc(
    __isl_take isl_set *set,
    __isl_take isl_qpolynomial *qp);
__isl_give isl_pw_qpolynomial *isl_pw_qpolynomial_from_qpolynomial(
    __isl_take isl_qpolynomial *qp);
__isl_give isl_pw_qpolynomial *isl_pw_qpolynomial_from_pw_aff(
    __isl_take isl_pw_aff *pwaff);

__isl_give isl_union_pw_qpolynomial *isl_union_pw_qpolynomial_zero(
    __isl_take isl_space *space);
__isl_give isl_union_pw_qpolynomial *isl_union_pw_qpolynomial_from_pw_qpolynomial(
    __isl_take isl_pw_qpolynomial *pwqp);
```

```

__isl_give isl_union_pw_qpolynomial *isl_union_pw_qpolynomial_add_pw_qpolynomial(
    __isl_take isl_union_pw_qpolynomial *upwqp,
    __isl_take isl_pw_qpolynomial *pwqp);

```

Quasipolynomials can be copied and freed again using the following functions.

```

__isl_give isl_qpolynomial *isl_qpolynomial_copy(
    __isl_keep isl_qpolynomial *qp);
__isl_null isl_qpolynomial *isl_qpolynomial_free(
    __isl_take isl_qpolynomial *qp);

__isl_give isl_pw_qpolynomial *isl_pw_qpolynomial_copy(
    __isl_keep isl_pw_qpolynomial *pwqp);
__isl_null isl_pw_qpolynomial *isl_pw_qpolynomial_free(
    __isl_take isl_pw_qpolynomial *pwqp);

__isl_give isl_union_pw_qpolynomial *isl_union_pw_qpolynomial_copy(
    __isl_keep isl_union_pw_qpolynomial *upwqp);
__isl_null isl_union_pw_qpolynomial *
isl_union_pw_qpolynomial_free(
    __isl_take isl_union_pw_qpolynomial *upwqp);

```

Inspecting (Piecewise) Quasipolynomials

To iterate over all piecewise quasipolynomials in a union piecewise quasipolynomial, use the following function

```

int isl_union_pw_qpolynomial_foreach_pw_qpolynomial(
    __isl_keep isl_union_pw_qpolynomial *upwqp,
    int (*fn)(__isl_take isl_pw_qpolynomial *pwqp, void *user),
    void *user);

```

To extract the piecewise quasipolynomial in a given space from a union, use

```

__isl_give isl_pw_qpolynomial *
isl_union_pw_qpolynomial_extract_pw_qpolynomial(
    __isl_keep isl_union_pw_qpolynomial *upwqp,
    __isl_take isl_space *space);

```

To iterate over the cells in a piecewise quasipolynomial, use either of the following two functions

```

int isl_pw_qpolynomial_foreach_piece(
    __isl_keep isl_pw_qpolynomial *pwqp,
    int (*fn)(__isl_take isl_set *set,
        __isl_take isl_qpolynomial *qp,
        void *user), void *user);
int isl_pw_qpolynomial_foreach_lifted_piece(

```

```

__isl_keep isl_pw_qpolynomial *pwqp,
int (*fn)(__isl_take isl_set *set,
__isl_take isl_qpolynomial *qp,
void *user), void *user);

```

As usual, the function `fn` should return 0 on success and -1 on failure. The difference between `isl_pw_qpolynomial_foreach_piece` and `isl_pw_qpolynomial_foreach_lifted_piece` is that `isl_pw_qpolynomial_foreach_lifted_piece` will first compute unique representations for all existentially quantified variables and then turn these existentially quantified variables into extra set variables, adapting the associated quasipolynomial accordingly. This means that the set passed to `fn` will not have any existentially quantified variables, but that the dimensions of the sets may be different for different invocations of `fn`.

The constant term of a quasipolynomial can be extracted using

```

__isl_give isl_val *isl_qpolynomial_get_constant_val(
__isl_keep isl_qpolynomial *qp);

```

To iterate over all terms in a quasipolynomial, use

```

int isl_qpolynomial_foreach_term(
__isl_keep isl_qpolynomial *qp,
int (*fn)(__isl_take isl_term *term,
void *user), void *user);

```

The terms themselves can be inspected and freed using these functions

```

unsigned isl_term_dim(__isl_keep isl_term *term,
enum isl_dim_type type);
__isl_give isl_val *isl_term_get_coefficient_val(
__isl_keep isl_term *term);
int isl_term_get_exp(__isl_keep isl_term *term,
enum isl_dim_type type, unsigned pos);
__isl_give isl_aff *isl_term_get_div(
__isl_keep isl_term *term, unsigned pos);
void isl_term_free(__isl_take isl_term *term);

```

Each term is a product of parameters, set variables and integer divisions. The function `isl_term_get_exp` returns the exponent of a given dimensions in the given term.

Properties of (Piecewise) Quasipolynomials

To check whether two union piecewise quasipolynomials are obviously equal, use

```

int isl_union_pw_qpolynomial_plain_is_equal(
__isl_keep isl_union_pw_qpolynomial *upwqp1,
__isl_keep isl_union_pw_qpolynomial *upwqp2);

```

Operations on (Piecewise) Quasipolynomials

```
__isl_give isl_qpolynomial *isl_qpolynomial_scale_val(
    __isl_take isl_qpolynomial *qp,
    __isl_take isl_val *v);
__isl_give isl_qpolynomial *isl_qpolynomial_neg(
    __isl_take isl_qpolynomial *qp);
__isl_give isl_qpolynomial *isl_qpolynomial_add(
    __isl_take isl_qpolynomial *qp1,
    __isl_take isl_qpolynomial *qp2);
__isl_give isl_qpolynomial *isl_qpolynomial_sub(
    __isl_take isl_qpolynomial *qp1,
    __isl_take isl_qpolynomial *qp2);
__isl_give isl_qpolynomial *isl_qpolynomial_mul(
    __isl_take isl_qpolynomial *qp1,
    __isl_take isl_qpolynomial *qp2);
__isl_give isl_qpolynomial *isl_qpolynomial_pow(
    __isl_take isl_qpolynomial *qp, unsigned exponent);

__isl_give isl_pw_qpolynomial *isl_pw_qpolynomial_fix_val(
    __isl_take isl_pw_qpolynomial *pwqp,
    enum isl_dim_type type, unsigned n,
    __isl_take isl_val *v);
__isl_give isl_pw_qpolynomial *
isl_pw_qpolynomial_scale_val(
    __isl_take isl_pw_qpolynomial *pwqp,
    __isl_take isl_val *v);
__isl_give isl_pw_qpolynomial *isl_pw_qpolynomial_add(
    __isl_take isl_pw_qpolynomial *pwqp1,
    __isl_take isl_pw_qpolynomial *pwqp2);
__isl_give isl_pw_qpolynomial *isl_pw_qpolynomial_sub(
    __isl_take isl_pw_qpolynomial *pwqp1,
    __isl_take isl_pw_qpolynomial *pwqp2);
__isl_give isl_pw_qpolynomial *isl_pw_qpolynomial_add_disjoint(
    __isl_take isl_pw_qpolynomial *pwqp1,
    __isl_take isl_pw_qpolynomial *pwqp2);
__isl_give isl_pw_qpolynomial *isl_pw_qpolynomial_neg(
    __isl_take isl_pw_qpolynomial *pwqp);
__isl_give isl_pw_qpolynomial *isl_pw_qpolynomial_mul(
    __isl_take isl_pw_qpolynomial *pwqp1,
    __isl_take isl_pw_qpolynomial *pwqp2);
__isl_give isl_pw_qpolynomial *isl_pw_qpolynomial_pow(
    __isl_take isl_pw_qpolynomial *pwqp, unsigned exponent);

__isl_give isl_union_pw_qpolynomial *
isl_union_pw_qpolynomial_scale_val(
    __isl_take isl_union_pw_qpolynomial *upwqp,
```

```

        __isl_take isl_val *v);
__isl_give isl_union_pw_qpolynomial *isl_union_pw_qpolynomial_add(
    __isl_take isl_union_pw_qpolynomial *upwqp1,
    __isl_take isl_union_pw_qpolynomial *upwqp2);
__isl_give isl_union_pw_qpolynomial *isl_union_pw_qpolynomial_sub(
    __isl_take isl_union_pw_qpolynomial *upwqp1,
    __isl_take isl_union_pw_qpolynomial *upwqp2);
__isl_give isl_union_pw_qpolynomial *isl_union_pw_qpolynomial_mul(
    __isl_take isl_union_pw_qpolynomial *upwqp1,
    __isl_take isl_union_pw_qpolynomial *upwqp2);

__isl_give isl_val *isl_pw_qpolynomial_eval(
    __isl_take isl_pw_qpolynomial *pwqp,
    __isl_take isl_point *pnt);

__isl_give isl_val *isl_union_pw_qpolynomial_eval(
    __isl_take isl_union_pw_qpolynomial *upwqp,
    __isl_take isl_point *pnt);

__isl_give isl_set *isl_pw_qpolynomial_domain(
    __isl_take isl_pw_qpolynomial *pwqp);
__isl_give isl_pw_qpolynomial *isl_pw_qpolynomial_intersect_domain(
    __isl_take isl_pw_qpolynomial *pwpq,
    __isl_take isl_set *set);
__isl_give isl_pw_qpolynomial *isl_pw_qpolynomial_intersect_params(
    __isl_take isl_pw_qpolynomial *pwpq,
    __isl_take isl_set *set);

__isl_give isl_union_set *isl_union_pw_qpolynomial_domain(
    __isl_take isl_union_pw_qpolynomial *upwqp);
__isl_give isl_union_pw_qpolynomial *isl_union_pw_qpolynomial_intersect_domain(
    __isl_take isl_union_pw_qpolynomial *upwqp,
    __isl_take isl_union_set *uset);
__isl_give isl_union_pw_qpolynomial *
isl_union_pw_qpolynomial_intersect_params(
    __isl_take isl_union_pw_qpolynomial *upwqp,
    __isl_take isl_set *set);

__isl_give isl_qpolynomial *isl_qpolynomial_align_params(
    __isl_take isl_qpolynomial *qp,
    __isl_take isl_space *model);

__isl_give isl_qpolynomial *isl_qpolynomial_project_domain_on_params(
    __isl_take isl_qpolynomial *qp);
__isl_give isl_pw_qpolynomial *isl_pw_qpolynomial_project_domain_on_params(
    __isl_take isl_pw_qpolynomial *pwqp);

```

```

__isl_give isl_union_pw_qpolynomial *isl_union_pw_qpolynomial_coalesce(
    __isl_take isl_union_pw_qpolynomial *upwqp);

__isl_give isl_qpolynomial *isl_qpolynomial_gist_params(
    __isl_take isl_qpolynomial *qp,
    __isl_take isl_set *context);
__isl_give isl_qpolynomial *isl_qpolynomial_gist(
    __isl_take isl_qpolynomial *qp,
    __isl_take isl_set *context);

__isl_give isl_pw_qpolynomial *isl_pw_qpolynomial_gist_params(
    __isl_take isl_pw_qpolynomial *pwqp,
    __isl_take isl_set *context);
__isl_give isl_pw_qpolynomial *isl_pw_qpolynomial_gist(
    __isl_take isl_pw_qpolynomial *pwqp,
    __isl_take isl_set *context);

__isl_give isl_union_pw_qpolynomial *
isl_union_pw_qpolynomial_gist_params(
    __isl_take isl_union_pw_qpolynomial *upwqp,
    __isl_take isl_set *context);
__isl_give isl_union_pw_qpolynomial *isl_union_pw_qpolynomial_gist(
    __isl_take isl_union_pw_qpolynomial *upwqp,
    __isl_take isl_union_set *context);

```

The gist operation applies the gist operation to each of the cells in the domain of the input piecewise quasipolynomial. The context is also exploited to simplify the quasipolynomials associated to each cell.

```

__isl_give isl_pw_qpolynomial *isl_pw_qpolynomial_to_polynomial(
    __isl_take isl_pw_qpolynomial *pwqp, int sign);
__isl_give isl_union_pw_qpolynomial *
isl_union_pw_qpolynomial_to_polynomial(
    __isl_take isl_union_pw_qpolynomial *upwqp, int sign);

```

Approximate each quasipolynomial by a polynomial. If `sign` is positive, the polynomial will be an overapproximation. If `sign` is negative, it will be an underapproximation. If `sign` is zero, the approximation will lie somewhere in between.

1.4.24 Bounds on Piecewise Quasipolynomials and Piecewise Quasipolynomial Reductions

A piecewise quasipolynomial reduction is a piecewise reduction (or fold) of quasipolynomials. In particular, the reduction can be maximum or a minimum. The objects are mainly used to represent the result of an upper or lower bound on a quasipolynomial over its domain, i.e., as the result of the following function.

```

__isl_give isl_pw_qpolynomial_fold *isl_pw_qpolynomial_bound(
    __isl_take isl_pw_qpolynomial *pwqp,
    enum isl_fold type, int *tight);

__isl_give isl_union_pw_qpolynomial_fold *isl_union_pw_qpolynomial_bound(
    __isl_take isl_union_pw_qpolynomial *upwqp,
    enum isl_fold type, int *tight);

```

The type argument may be either `isl_fold_min` or `isl_fold_max`. If `tight` is not NULL, then `*tight` is set to 1 if the returned bound is known to be tight, i.e., for each value of the parameters there is at least one element in the domain that reaches the bound. If the domain of `pwqp` is not wrapping, then the bound is computed over all elements in that domain and the result has a purely parametric domain. If the domain of `pwqp` is wrapping, then the bound is computed over the range of the wrapped relation. The domain of the wrapped relation becomes the domain of the result.

A (piecewise) quasipolynomial reduction can be copied or freed using the following functions.

```

__isl_give isl_qpolynomial_fold *isl_qpolynomial_fold_copy(
    __isl_keep isl_qpolynomial_fold *fold);
__isl_give isl_pw_qpolynomial_fold *isl_pw_qpolynomial_fold_copy(
    __isl_keep isl_pw_qpolynomial_fold *pwf);
__isl_give isl_union_pw_qpolynomial_fold *isl_union_pw_qpolynomial_fold_copy(
    __isl_keep isl_union_pw_qpolynomial_fold *upwf);
void isl_qpolynomial_fold_free(
    __isl_take isl_qpolynomial_fold *fold);
__isl_null isl_pw_qpolynomial_fold *
isl_pw_qpolynomial_fold_free(
    __isl_take isl_pw_qpolynomial_fold *pwf);
__isl_null isl_union_pw_qpolynomial_fold *
isl_union_pw_qpolynomial_fold_free(
    __isl_take isl_union_pw_qpolynomial_fold *upwf);

```

Printing Piecewise Quasipolynomial Reductions

Piecewise quasipolynomial reductions can be printed using the following function.

```

__isl_give isl_printer *isl_printer_print_pw_qpolynomial_fold(
    __isl_take isl_printer *p,
    __isl_keep isl_pw_qpolynomial_fold *pwf);
__isl_give isl_printer *isl_printer_print_union_pw_qpolynomial_fold(
    __isl_take isl_printer *p,
    __isl_keep isl_union_pw_qpolynomial_fold *upwf);

```

For `isl_printer_print_pw_qpolynomial_fold`, output format of the printer needs to be set to either `ISL_FORMAT_ISL` or `ISL_FORMAT_C`. For `isl_printer_print_union_pw_qpolynomial_fold`, output format of the printer needs to be set to `ISL_FORMAT_ISL`. In case of printing in `ISL_FORMAT_C`, the user may want to set the names of all dimensions


```

__isl_give isl_pw_qpolynomial_fold *
isl_pw_qpolynomial_fold_set_dim_name(
    __isl_take isl_pw_qpolynomial_fold *pwf,
    enum isl_dim_type type, unsigned pos,
    const char *s);

```

Inspecting (Piecewise) Quasipolynomial Reductions

To iterate over all piecewise quasipolynomial reductions in a union piecewise quasipolynomial reduction, use the following function

```

int isl_union_pw_qpolynomial_fold_foreach_pw_qpolynomial_fold(
    __isl_keep isl_union_pw_qpolynomial_fold *upwf,
    int (*fn)(__isl_take isl_pw_qpolynomial_fold *pwf,
        void *user), void *user);

```

To iterate over the cells in a piecewise quasipolynomial reduction, use either of the following two functions

```

int isl_pw_qpolynomial_fold_foreach_piece(
    __isl_keep isl_pw_qpolynomial_fold *pwf,
    int (*fn)(__isl_take isl_set *set,
        __isl_take isl_qpolynomial_fold *fold,
        void *user), void *user);
int isl_pw_qpolynomial_fold_foreach_lifted_piece(
    __isl_keep isl_pw_qpolynomial_fold *pwf,
    int (*fn)(__isl_take isl_set *set,
        __isl_take isl_qpolynomial_fold *fold,
        void *user), void *user);

```

See Inspecting (Piecewise) Quasipolynomials for an explanation of the difference between these two functions.

To iterate over all quasipolynomials in a reduction, use

```

int isl_qpolynomial_fold_foreach_qpolynomial(
    __isl_keep isl_qpolynomial_fold *fold,
    int (*fn)(__isl_take isl_qpolynomial *qp,
        void *user), void *user);

```

Properties of Piecewise Quasipolynomial Reductions

To check whether two union piecewise quasipolynomial reductions are obviously equal, use

```

int isl_union_pw_qpolynomial_fold_plain_is_equal(
    __isl_keep isl_union_pw_qpolynomial_fold *upwf1,
    __isl_keep isl_union_pw_qpolynomial_fold *upwf2);

```

Operations on Piecewise Quasipolynomial Reductions

```
__isl_give isl_qpolynomial_fold *isl_qpolynomial_fold_scale_val(
    __isl_take isl_qpolynomial_fold *fold,
    __isl_take isl_val *v);
__isl_give isl_pw_qpolynomial_fold *
isl_pw_qpolynomial_fold_scale_val(
    __isl_take isl_pw_qpolynomial_fold *pwf,
    __isl_take isl_val *v);
__isl_give isl_union_pw_qpolynomial_fold *
isl_union_pw_qpolynomial_fold_scale_val(
    __isl_take isl_union_pw_qpolynomial_fold *upwf,
    __isl_take isl_val *v);

__isl_give isl_pw_qpolynomial_fold *isl_pw_qpolynomial_fold_add(
    __isl_take isl_pw_qpolynomial_fold *pwf1,
    __isl_take isl_pw_qpolynomial_fold *pwf2);

__isl_give isl_pw_qpolynomial_fold *isl_pw_qpolynomial_fold_fold(
    __isl_take isl_pw_qpolynomial_fold *pwf1,
    __isl_take isl_pw_qpolynomial_fold *pwf2);

__isl_give isl_union_pw_qpolynomial_fold *isl_union_pw_qpolynomial_fold_fold(
    __isl_take isl_union_pw_qpolynomial_fold *upwf1,
    __isl_take isl_union_pw_qpolynomial_fold *upwf2);

__isl_give isl_val *isl_pw_qpolynomial_fold_eval(
    __isl_take isl_pw_qpolynomial_fold *pwf,
    __isl_take isl_point *pnt);

__isl_give isl_val *isl_union_pw_qpolynomial_fold_eval(
    __isl_take isl_union_pw_qpolynomial_fold *upwf,
    __isl_take isl_point *pnt);

__isl_give isl_pw_qpolynomial_fold *
isl_pw_qpolynomial_fold_intersect_params(
    __isl_take isl_pw_qpolynomial_fold *pwf,
    __isl_take isl_set *set);

__isl_give isl_union_set *isl_union_pw_qpolynomial_fold_domain(
    __isl_take isl_union_pw_qpolynomial_fold *upwf);
__isl_give isl_union_pw_qpolynomial_fold *isl_union_pw_qpolynomial_fold_intersect_d
    __isl_take isl_union_pw_qpolynomial_fold *upwf,
    __isl_take isl_union_set *uset);
__isl_give isl_union_pw_qpolynomial_fold *
isl_union_pw_qpolynomial_fold_intersect_params(
    __isl_take isl_union_pw_qpolynomial_fold *upwf,
    __isl_take isl_set *set);
```

```

__isl_give isl_pw_qpolynomial_fold *isl_pw_qpolynomial_fold_project_domain_on_params(
    __isl_take isl_pw_qpolynomial_fold *pwf);

__isl_give isl_pw_qpolynomial_fold *isl_pw_qpolynomial_fold_coalesce(
    __isl_take isl_pw_qpolynomial_fold *pwf);

__isl_give isl_union_pw_qpolynomial_fold *isl_union_pw_qpolynomial_fold_coalesce(
    __isl_take isl_union_pw_qpolynomial_fold *upwf);

__isl_give isl_qpolynomial_fold *isl_qpolynomial_fold_gist_params(
    __isl_take isl_qpolynomial_fold *fold,
    __isl_take isl_set *context);
__isl_give isl_qpolynomial_fold *isl_qpolynomial_fold_gist(
    __isl_take isl_qpolynomial_fold *fold,
    __isl_take isl_set *context);

__isl_give isl_pw_qpolynomial_fold *isl_pw_qpolynomial_fold_gist(
    __isl_take isl_pw_qpolynomial_fold *pwf,
    __isl_take isl_set *context);
__isl_give isl_pw_qpolynomial_fold *isl_pw_qpolynomial_fold_gist_params(
    __isl_take isl_pw_qpolynomial_fold *pwf,
    __isl_take isl_set *context);

__isl_give isl_union_pw_qpolynomial_fold *isl_union_pw_qpolynomial_fold_gist(
    __isl_take isl_union_pw_qpolynomial_fold *upwf,
    __isl_take isl_union_set *context);
__isl_give isl_union_pw_qpolynomial_fold *
isl_union_pw_qpolynomial_fold_gist_params(
    __isl_take isl_union_pw_qpolynomial_fold *upwf,
    __isl_take isl_set *context);

```

The gist operation applies the gist operation to each of the cells in the domain of the input piecewise quasipolynomial reduction. In future, the operation will also exploit the context to simplify the quasipolynomial reductions associated to each cell.

```

__isl_give isl_pw_qpolynomial_fold *
isl_set_apply_pw_qpolynomial_fold(
    __isl_take isl_set *set,
    __isl_take isl_pw_qpolynomial_fold *pwf,
    int *tight);
__isl_give isl_pw_qpolynomial_fold *
isl_map_apply_pw_qpolynomial_fold(
    __isl_take isl_map *map,
    __isl_take isl_pw_qpolynomial_fold *pwf,
    int *tight);
__isl_give isl_union_pw_qpolynomial_fold *
isl_union_set_apply_union_pw_qpolynomial_fold(

```

```

        __isl_take isl_union_set *uset,
        __isl_take isl_union_pw_qpolynomial_fold *upwf,
        int *tight);
__isl_give isl_union_pw_qpolynomial_fold *
isl_union_map_apply_union_pw_qpolynomial_fold(
    __isl_take isl_union_map *umap,
    __isl_take isl_union_pw_qpolynomial_fold *upwf,
    int *tight);

```

The functions taking a map compose the given map with the given piecewise quasipolynomial reduction. That is, compute a bound (of the same type as `pwf` or `upwf` itself) over all elements in the intersection of the range of the map and the domain of the piecewise quasipolynomial reduction as a function of an element in the domain of the map. The functions taking a set compute a bound over all elements in the intersection of the set and the domain of the piecewise quasipolynomial reduction.

1.4.25 Parametric Vertex Enumeration

The parametric vertex enumeration described in this section is mainly intended to be used internally and by the `barvinok` library.

```

#include <isl/vertices.h>
__isl_give isl_vertices *isl_basic_set_compute_vertices(
    __isl_keep isl_basic_set *bset);

```

The function `isl_basic_set_compute_vertices` performs the actual computation of the parametric vertices and the chamber decomposition and store the result in an `isl_vertices` object. This information can be queried by either iterating over all the vertices or iterating over all the chambers or cells and then iterating over all vertices that are active on the chamber.

```

int isl_vertices_foreach_vertex(
    __isl_keep isl_vertices *vertices,
    int (*fn)(__isl_take isl_vertex *vertex, void *user),
    void *user);

int isl_vertices_foreach_cell(
    __isl_keep isl_vertices *vertices,
    int (*fn)(__isl_take isl_cell *cell, void *user),
    void *user);

int isl_cell_foreach_vertex(__isl_keep isl_cell *cell,
    int (*fn)(__isl_take isl_vertex *vertex, void *user),
    void *user);

```

Other operations that can be performed on an `isl_vertices` object are the following.

```
isl_ctx *isl_vertices_get_ctx(
    __isl_keep isl_vertices *vertices);
int isl_vertices_get_n_vertices(
    __isl_keep isl_vertices *vertices);
void isl_vertices_free(__isl_take isl_vertices *vertices);
```

Vertices can be inspected and destroyed using the following functions.

```
isl_ctx *isl_vertex_get_ctx(__isl_keep isl_vertex *vertex);
int isl_vertex_get_id(__isl_keep isl_vertex *vertex);
__isl_give isl_basic_set *isl_vertex_get_domain(
    __isl_keep isl_vertex *vertex);
__isl_give isl_multi_aff *isl_vertex_get_expr(
    __isl_keep isl_vertex *vertex);
void isl_vertex_free(__isl_take isl_vertex *vertex);
```

`isl_vertex_get_expr` returns a multiple quasi-affine expression describing the vertex in terms of the parameters, while `isl_vertex_get_domain` returns the activity domain of the vertex.

Chambers can be inspected and destroyed using the following functions.

```
isl_ctx *isl_cell_get_ctx(__isl_keep isl_cell *cell);
__isl_give isl_basic_set *isl_cell_get_domain(
    __isl_keep isl_cell *cell);
void isl_cell_free(__isl_take isl_cell *cell);
```

1.5 Polyhedral Compilation Library

This section collects functionality in `isl` that has been specifically designed for use during polyhedral compilation.

1.5.1 Dependence Analysis

`isl` contains specialized functionality for performing array dataflow analysis. That is, given a *sink* access relation and a collection of possible *source* access relations, `isl` can compute relations that describe for each iteration of the sink access, which iteration of which of the source access relations was the last to access the same data element before the given iteration of the sink access. The resulting dependence relations map source iterations to the corresponding sink iterations. To compute standard flow dependences, the sink should be a read, while the sources should be writes. If any of the source accesses are marked as being *may* accesses, then there will be a dependence from the last *must* access **and** from any *may* access that follows this last *must* access. In particular, if *all* sources are *may* accesses, then memory based dependence analysis is performed. If, on the other hand, all sources are *must* accesses, then value based dependence analysis is performed.

```
#include <isl/flow.h>
```

```

typedef int (*isl_access_level_before)(void *first, void *second);

__isl_give isl_access_info *isl_access_info_alloc(
    __isl_take isl_map *sink,
    void *sink_user, isl_access_level_before fn,
    int max_source);
__isl_give isl_access_info *isl_access_info_add_source(
    __isl_take isl_access_info *acc,
    __isl_take isl_map *source, int must,
    void *source_user);
__isl_null isl_access_info *isl_access_info_free(
    __isl_take isl_access_info *acc);

__isl_give isl_flow *isl_access_info_compute_flow(
    __isl_take isl_access_info *acc);

int isl_flow_foreach(__isl_keep isl_flow *deps,
    int (*fn)(__isl_take isl_map *dep, int must,
        void *dep_user, void *user),
    void *user);
__isl_give isl_map *isl_flow_get_no_source(
    __isl_keep isl_flow *deps, int must);
void isl_flow_free(__isl_take isl_flow *deps);

```

The function `isl_access_info_compute_flow` performs the actual dependence analysis. The other functions are used to construct the input for this function or to read off the output.

The input is collected in an `isl_access_info`, which can be created through a call to `isl_access_info_alloc`. The arguments to this functions are the sink access relation `sink`, a token `sink_user` used to identify the sink access to the user, a callback function for specifying the relative order of source and sink accesses, and the number of source access relations that will be added. The callback function has type `int (*)(void *first, void *second)`. The function is called with two user supplied tokens identifying either a source or the sink and it should return the shared nesting level and the relative order of the two accesses. In particular, let n be the number of loops shared by the two accesses. If `first` precedes `second` textually, then the function should return $2 * n + 1$; otherwise, it should return $2 * n$. The sources can be added to the `isl_access_info` by performing (at most) `max_source` calls to `isl_access_info.add_source`. `must` indicates whether the source is a *must* access or a *may* access. Note that a multi-valued access relation should only be marked *must* if every iteration in the domain of the relation accesses *all* elements in its image. The `source_user` token is again used to identify the source access. The range of the source access relation `source` should have the same dimension as the range of the sink access relation. The `isl_access_info.free` function should usually not be called explicitly, because it is called implicitly by `isl_access_info.compute_flow`.

The result of the dependence analysis is collected in an `isl_flow`. There may be elements of the sink access for which no preceding source access could be found or for

which all preceding sources are *may* accesses. The relations containing these elements can be obtained through calls to `isl_flow_get_no_source`, the first with `must` set and the second with `must` unset. In the case of standard flow dependence analysis, with the sink a read and the sources *must* writes, the first relation corresponds to the reads from uninitialized array elements and the second relation is empty. The actual flow dependences can be extracted using `isl_flow_foreach`. This function will call the user-specified callback function `fn` for each **non-empty** dependence between a source and the sink. The callback function is called with four arguments, the actual flow dependence relation mapping source iterations to sink iterations, a boolean that indicates whether it is a *must* or *may* dependence, a token identifying the source and an additional `void *` with value equal to the third argument of the `isl_flow_foreach` call. A dependence is marked *must* if it originates from a *must* source and if it is not followed by any *may* sources.

After finishing with an `isl_flow`, the user should call `isl_flow_free` to free all associated memory.

A higher-level interface to dependence analysis is provided by the following function.

```
#include <isl/flow.h>

int isl_union_map_compute_flow(__isl_take isl_union_map *sink,
                              __isl_take isl_union_map *must_source,
                              __isl_take isl_union_map *may_source,
                              __isl_take isl_union_map *schedule,
                              __isl_give isl_union_map **must_dep,
                              __isl_give isl_union_map **may_dep,
                              __isl_give isl_union_map **must_no_source,
                              __isl_give isl_union_map **may_no_source);
```

The arrays are identified by the tuple names of the ranges of the accesses. The iteration domains by the tuple names of the domains of the accesses and of the schedule. The relative order of the iteration domains is given by the schedule. The relations returned through `must_no_source` and `may_no_source` are subsets of `sink`. Any of `must_dep`, `may_dep`, `must_no_source` or `may_no_source` may be `NULL`, but a `NULL` value for any of the other arguments is treated as an error.

Interaction with Dependence Analysis

During the dependence analysis, we frequently need to perform the following operation. Given a relation between sink iterations and potential source iterations from a particular source domain, what is the last potential source iteration corresponding to each sink iteration. It can sometimes be convenient to adjust the set of potential source iterations before or after each such operation. The prototypical example is fuzzy array dataflow analysis, where we need to analyze if, based on data-dependent constraints, the sink iteration can ever be executed without one or more of the corresponding potential source iterations being executed. If so, we can introduce extra parameters and

select an unknown but fixed source iteration from the potential source iterations. To be able to perform such manipulations, isl provides the following function.

```
#include <isl/flow.h>

typedef __isl_give isl_restriction *(*isl_access_restrict)(
    __isl_keep isl_map *source_map,
    __isl_keep isl_set *sink, void *source_user,
    void *user);
__isl_give isl_access_info *isl_access_info_set_restrict(
    __isl_take isl_access_info *acc,
    isl_access_restrict fn, void *user);
```

The function `isl_access_info_set_restrict` should be called before calling `isl_access_info_compute_flow` and registers a callback function that will be called any time isl is about to compute the last potential source. The first argument is the (reverse) proto-dependence, mapping sink iterations to potential source iterations. The second argument represents the sink iterations for which we want to compute the last source iteration. The third argument is the token corresponding to the source and the final argument is the token passed to `isl_access_info_set_restrict`. The callback is expected to return a restriction on either the input or the output of the operation computing the last potential source. If the input needs to be restricted then restrictions are needed for both the source and the sink iterations. The sink iterations and the potential source iterations will be intersected with these sets. If the output needs to be restricted then only a restriction on the source iterations is required. If any error occurs, the callback should return NULL. An `isl_restriction` object can be created, freed and inspected using the following functions.

```
#include <isl/flow.h>

__isl_give isl_restriction *isl_restriction_input(
    __isl_take isl_set *source_restr,
    __isl_take isl_set *sink_restr);
__isl_give isl_restriction *isl_restriction_output(
    __isl_take isl_set *source_restr);
__isl_give isl_restriction *isl_restriction_none(
    __isl_take isl_map *source_map);
__isl_give isl_restriction *isl_restriction_empty(
    __isl_take isl_map *source_map);
__isl_null isl_restriction *isl_restriction_free(
    __isl_take isl_restriction *restr);
isl_ctx *isl_restriction_get_ctx(
    __isl_keep isl_restriction *restr);
```

`isl_restriction_none` and `isl_restriction_empty` are special cases of `isl_restriction_input`. `isl_restriction_none` is essentially equivalent to


```
isl_restriction_input(isl_set_universe(
    isl_space_range(isl_map_get_space(source_map))),
    isl_set_universe(
    isl_space_domain(isl_map_get_space(source_map))));
```

whereas `isl_restriction_empty` is essentially equivalent to

```
isl_restriction_input(isl_set_empty(
    isl_space_range(isl_map_get_space(source_map))),
    isl_set_universe(
    isl_space_domain(isl_map_get_space(source_map))));
```

1.5.2 Scheduling

The functionality described in this section is fairly new and may be subject to change.

```
#include <isl/schedule.h>
__isl_give isl_schedule *
isl_schedule_constraints_compute_schedule(
    __isl_take isl_schedule_constraints *sc);
__isl_null isl_schedule *isl_schedule_free(
    __isl_take isl_schedule *sched);
```

The function `isl_schedule_constraints_compute_schedule` can be used to compute a schedule that satisfies the given schedule constraints. These schedule constraints include the iteration domain for which a schedule should be computed and dependences between pairs of iterations. In particular, these dependences include *validity* dependences and *proximity* dependences. By default, the algorithm used to construct the schedule is similar to that of Pluto. Alternatively, Feautrier's multi-dimensional scheduling algorithm can be selected. The generated schedule respects all validity dependences. That is, all dependence distances over these dependences in the scheduled space are lexicographically positive. The default algorithm tries to ensure that the dependence distances over coincidence constraints are zero and to minimize the dependence distances over proximity dependences. Moreover, it tries to obtain sequences (bands) of schedule dimensions for groups of domains where the dependence distances over validity dependences have only non-negative values. When using Feautrier's algorithm, the coincidence and proximity constraints are only taken into account during the extension to a full-dimensional schedule.

An `isl_schedule_constraints` object can be constructed and manipulated using the following functions.

```
#include <isl/schedule.h>
__isl_give isl_schedule_constraints *
isl_schedule_constraints_copy(
    __isl_keep isl_schedule_constraints *sc);
__isl_give isl_schedule_constraints *
```

```

isl_schedule_constraints_on_domain(
    __isl_take isl_union_set *domain);
isl_ctx *isl_schedule_constraints_get_ctx(
    __isl_keep isl_schedule_constraints *sc);
__isl_give isl_schedule_constraints *
isl_schedule_constraints_set_validity(
    __isl_take isl_schedule_constraints *sc,
    __isl_take isl_union_map *validity);
__isl_give isl_schedule_constraints *
isl_schedule_constraints_set_coincidence(
    __isl_take isl_schedule_constraints *sc,
    __isl_take isl_union_map *coincidence);
__isl_give isl_schedule_constraints *
isl_schedule_constraints_set_proximity(
    __isl_take isl_schedule_constraints *sc,
    __isl_take isl_union_map *proximity);
__isl_give isl_schedule_constraints *
isl_schedule_constraints_set_conditional_validity(
    __isl_take isl_schedule_constraints *sc,
    __isl_take isl_union_map *condition,
    __isl_take isl_union_map *validity);
__isl_null isl_schedule_constraints *
isl_schedule_constraints_free(
    __isl_take isl_schedule_constraints *sc);

```

The initial `isl_schedule_constraints` object created by `isl_schedule_constraints_on_domain` does not impose any constraints. That is, it has an empty set of dependences. The function `isl_schedule_constraints_set_validity` replaces the validity dependences, mapping domain elements i to domain elements that should be scheduled after i . The function `isl_schedule_constraints_set_coincidence` replaces the coincidence dependences, mapping domain elements i to domain elements that should be scheduled together with i , if possible. The function `isl_schedule_constraints_set_proximity` replaces the proximity dependences, mapping domain elements i to domain elements that should be scheduled either before i or as early as possible after i .

The function `isl_schedule_constraints_set_conditional_validity` replaces the conditional validity constraints. A conditional validity constraint is only imposed when any of the corresponding conditions is satisfied, i.e., when any of them is non-zero. That is, the scheduler ensures that within each band if the dependence distances over the condition constraints are not all zero then all corresponding conditional validity constraints are respected. A conditional validity constraint corresponds to a condition if the two are adjacent, i.e., if the domain of one relation intersect the range of the other relation. The typical use case of conditional validity constraints is to allow order constraints between live ranges to be violated as long as the live ranges themselves are local to the band. To allow more fine-grained control over which conditions correspond to which conditional validity constraints, the domains and ranges of these relations may include *tags*. That is, the domains and ranges of those relation may them-

selves be wrapped relations where the iteration domain appears in the domain of those wrapped relations and the range of the wrapped relations can be arbitrarily chosen by the user. Conditions and conditional validity constraints are only considered adjacent to each other if the entire wrapped relation matches. In particular, a relation with a tag will never be considered adjacent to a relation without a tag.

The following function computes a schedule directly from an iteration domain and validity and proximity dependences and is implemented in terms of the functions described above. The use of `isl_union_set_compute_schedule` is discouraged.

```
#include <isl/schedule.h>
__isl_give isl_schedule *isl_union_set_compute_schedule(
    __isl_take isl_union_set *domain,
    __isl_take isl_union_map *validity,
    __isl_take isl_union_map *proximity);
```

A mapping from the domains to the scheduled space can be obtained from an `isl_schedule` using the following function.

```
__isl_give isl_union_map *isl_schedule_get_map(
    __isl_keep isl_schedule *sched);
```

A representation of the schedule can be printed using

```
__isl_give isl_printer *isl_printer_print_schedule(
    __isl_take isl_printer *p,
    __isl_keep isl_schedule *schedule);
```

A representation of the schedule as a forest of bands can be obtained using the following function.

```
__isl_give isl_band_list *isl_schedule_get_band_forest(
    __isl_keep isl_schedule *schedule);
```

The individual bands can be visited in depth-first post-order using the following function.

```
#include <isl/schedule.h>
int isl_schedule_foreach_band(
    __isl_keep isl_schedule *sched,
    int (*fn)(__isl_keep isl_band *band, void *user),
    void *user);
```

The list can be manipulated as explained in §1.4.15. The bands inside the list can be copied and freed using the following functions.

```
#include <isl/band.h>
__isl_give isl_band *isl_band_copy(
    __isl_keep isl_band *band);
__isl_null isl_band *isl_band_free(
    __isl_take isl_band *band);
```

Each band contains zero or more scheduling dimensions. These are referred to as the members of the band. The section of the schedule that corresponds to the band is referred to as the partial schedule of the band. For those nodes that participate in a band, the outer scheduling dimensions form the prefix schedule, while the inner scheduling dimensions form the suffix schedule. That is, if we take a cut of the band forest, then the union of the concatenations of the prefix, partial and suffix schedules of each band in the cut is equal to the entire schedule (modulo some possible padding at the end with zero scheduling dimensions). The properties of a band can be inspected using the following functions.

```
#include <isl/band.h>
isl_ctx *isl_band_get_ctx(__isl_keep isl_band *band);

int isl_band_has_children(__isl_keep isl_band *band);
__isl_give isl_band_list *isl_band_get_children(
    __isl_keep isl_band *band);

__isl_give isl_union_map *isl_band_get_prefix_schedule(
    __isl_keep isl_band *band);
__isl_give isl_union_map *isl_band_get_partial_schedule(
    __isl_keep isl_band *band);
__isl_give isl_union_map *isl_band_get_suffix_schedule(
    __isl_keep isl_band *band);

int isl_band_n_member(__isl_keep isl_band *band);
int isl_band_member_is_coincident(
    __isl_keep isl_band *band, int pos);

int isl_band_list_foreach_band(
    __isl_keep isl_band_list *list,
    int (*fn)(__isl_keep isl_band *band, void *user),
    void *user);
```

Note that a scheduling dimension is considered to be “coincident” if it satisfies the coincidence constraints within its band. That is, if the dependence distances of the coincidence constraints are all zero in that direction (for fixed iterations of outer bands). Like `isl_schedule_foreach_band`, the function `isl_band_list_foreach_band` calls `fn` on the bands in depth-first post-order.

A band can be tiled using the following function.

```
#include <isl/band.h>
int isl_band_tile(__isl_keep isl_band *band,
    __isl_take isl_vec *sizes);

int isl_options_set_tile_scale_tile_loops(isl_ctx *ctx,
    int val);
int isl_options_get_tile_scale_tile_loops(isl_ctx *ctx);
int isl_options_set_tile_shift_point_loops(isl_ctx *ctx,
    int val);
int isl_options_get_tile_shift_point_loops(isl_ctx *ctx);
```

The `isl_band_tile` function tiles the band using the given tile sizes inside its schedule. A new child band is created to represent the point loops and it is inserted between the modified band and its children. The `tile_scale_tile_loops` option specifies whether the tile loops iterators should be scaled by the tile sizes. If the `tile_shift_point_loops` option is set, then the point loops are shifted to start at zero.

A band can be split into two nested bands using the following function.

```
int isl_band_split(__isl_keep isl_band *band, int pos);
```

The resulting outer band contains the first `pos` dimensions of `band` while the inner band contains the remaining dimensions.

A representation of the band can be printed using

```
#include <isl/band.h>
__isl_give isl_printer *isl_printer_print_band(
    __isl_take isl_printer *p,
    __isl_keep isl_band *band);
```

Options

```
#include <isl/schedule.h>
int isl_options_set_schedule_max_coefficient(
    isl_ctx *ctx, int val);
int isl_options_get_schedule_max_coefficient(
    isl_ctx *ctx);
int isl_options_set_schedule_max_constant_term(
    isl_ctx *ctx, int val);
int isl_options_get_schedule_max_constant_term(
    isl_ctx *ctx);
int isl_options_set_schedule_fuse(isl_ctx *ctx, int val);
int isl_options_get_schedule_fuse(isl_ctx *ctx);
int isl_options_set_schedule_maximize_band_depth(
    isl_ctx *ctx, int val);
int isl_options_get_schedule_maximize_band_depth(
    isl_ctx *ctx);
int isl_options_set_schedule_outer_coincidence(
    isl_ctx *ctx, int val);
int isl_options_get_schedule_outer_coincidence(
    isl_ctx *ctx);
int isl_options_set_schedule_split_scaled(
    isl_ctx *ctx, int val);
int isl_options_get_schedule_split_scaled(
    isl_ctx *ctx);
int isl_options_set_schedule_algorithm(
    isl_ctx *ctx, int val);
int isl_options_get_schedule_algorithm(
```

```

        isl_ctx *ctx);
int isl_options_set_schedule_separate_components(
    isl_ctx *ctx, int val);
int isl_options_get_schedule_separate_components(
    isl_ctx *ctx);

```

- `schedule_max_coefficient`

This option enforces that the coefficients for variable and parameter dimensions in the calculated schedule are not larger than the specified value. This option can significantly increase the speed of the scheduling calculation and may also prevent fusing of unrelated dimensions. A value of -1 means that this option does not introduce bounds on the variable or parameter coefficients.

- `schedule_max_constant_term`

This option enforces that the constant coefficients in the calculated schedule are not larger than the maximal constant term. This option can significantly increase the speed of the scheduling calculation and may also prevent fusing of unrelated dimensions. A value of -1 means that this option does not introduce bounds on the constant coefficients.

- `schedule_fuse`

This option controls the level of fusion. If this option is set to `ISL_SCHEDULE_FUSE_MIN`, then loops in the resulting schedule will be distributed as much as possible. If this option is set to `ISL_SCHEDULE_FUSE_MAX`, then `isl` will try to fuse loops in the resulting schedule.

- `schedule_maximize_band_depth`

If this option is set, we do not split bands at the point where we detect splitting is necessary. Instead, we backtrack and split bands as early as possible. This reduces the number of splits and maximizes the width of the bands. Wider bands give more possibilities for tiling. Note that if the `schedule_fuse` option is set to `ISL_SCHEDULE_FUSE_MIN`, then bands will be split as early as possible, even if there is no need. The `schedule_maximize_band_depth` option therefore has no effect in this case.

- `schedule_outer_coincidence`

If this option is set, then we try to construct schedules where the outermost scheduling dimension in each band satisfies the coincidence constraints.

- `schedule_split_scaled`

If this option is set, then we try to construct schedules in which the constant term is split off from the linear part if the linear parts of the scheduling rows for all nodes in the graphs have a common non-trivial divisor. The constant term is then placed in a separate band and the linear part is reduced.

- `schedule_algorithm`
Selects the scheduling algorithm to be used. Available scheduling algorithms are `ISL_SCHEDULE_ALGORITHM_ISL` and `ISL_SCHEDULE_ALGORITHM_FEAUTRIER`.
- `schedule_separate_components`
If at any point the dependence graph contains any (weakly connected) components, then these components are scheduled separately. If this option is not set, then some iterations of the domains in these components may be scheduled together. If this option is set, then the components are given consecutive schedules.

1.5.3 AST Generation

This section describes the `isl` functionality for generating ASTs that visit all the elements in a domain in an order specified by a schedule. In particular, given a `isl_union_map`, an AST is generated that visits all the elements in the domain of the `isl_union_map` according to the lexicographic order of the corresponding image element(s). If the range of the `isl_union_map` consists of elements in more than one space, then each of these spaces is handled separately in an arbitrary order. It should be noted that the image elements only specify the *order* in which the corresponding domain elements should be visited. No direct relation between the image elements and the loop iterators in the generated AST should be assumed.

Each AST is generated within a build. The initial build simply specifies the constraints on the parameters (if any) and can be created, inspected, copied and freed using the following functions.

```
#include <isl/ast_build.h>
__isl_give isl_ast_build *isl_ast_build_from_context(
    __isl_take isl_set *set);
isl_ctx *isl_ast_build_get_ctx(
    __isl_keep isl_ast_build *build);
__isl_give isl_ast_build *isl_ast_build_copy(
    __isl_keep isl_ast_build *build);
__isl_null isl_ast_build *isl_ast_build_free(
    __isl_take isl_ast_build *build);
```

The `set` argument is usually a parameter set with zero or more parameters. More `isl_ast_build` functions are described in §1.5.3 and §1.5.3. Finally, the AST itself can be constructed using the following function.

```
#include <isl/ast_build.h>
__isl_give isl_ast_node *isl_ast_build_ast_from_schedule(
    __isl_keep isl_ast_build *build,
    __isl_take isl_union_map *schedule);
```

Inspecting the AST

The basic properties of an AST node can be obtained as follows.

```
#include <isl/ast.h>
isl_ctx *isl_ast_node_get_ctx(
    __isl_keep isl_ast_node *node);
enum isl_ast_node_type isl_ast_node_get_type(
    __isl_keep isl_ast_node *node);
```

The type of an AST node is one of `isl_ast_node_for`, `isl_ast_node_if`, `isl_ast_node_block` or `isl_ast_node_user`. An `isl_ast_node_for` represents a for node. An `isl_ast_node_if` represents an if node. An `isl_ast_node_block` represents a compound node. An `isl_ast_node_user` represents an expression statement. An expression statement typically corresponds to a domain element, i.e., one of the elements that is visited by the AST.

Each type of node has its own additional properties.

```
#include <isl/ast.h>
__isl_give isl_ast_expr *isl_ast_node_for_get_iterator(
    __isl_keep isl_ast_node *node);
__isl_give isl_ast_expr *isl_ast_node_for_get_init(
    __isl_keep isl_ast_node *node);
__isl_give isl_ast_expr *isl_ast_node_for_get_cond(
    __isl_keep isl_ast_node *node);
__isl_give isl_ast_expr *isl_ast_node_for_get_inc(
    __isl_keep isl_ast_node *node);
__isl_give isl_ast_node *isl_ast_node_for_get_body(
    __isl_keep isl_ast_node *node);
int isl_ast_node_for_is_degenerate(
    __isl_keep isl_ast_node *node);
```

An `isl_ast_for` is considered degenerate if it is known to execute exactly once.

```
#include <isl/ast.h>
__isl_give isl_ast_expr *isl_ast_node_if_get_cond(
    __isl_keep isl_ast_node *node);
__isl_give isl_ast_node *isl_ast_node_if_get_then(
    __isl_keep isl_ast_node *node);
int isl_ast_node_if_has_else(
    __isl_keep isl_ast_node *node);
__isl_give isl_ast_node *isl_ast_node_if_get_else(
    __isl_keep isl_ast_node *node);

__isl_give isl_ast_node_list *
isl_ast_node_block_get_children(
    __isl_keep isl_ast_node *node);
```



```
__isl_give isl_ast_expr *isl_ast_node_user_get_expr(
    __isl_keep isl_ast_node *node);
```

Each of the returned `isl_ast_expr`s can in turn be inspected using the following functions.

```
#include <isl/ast.h>
isl_ctx *isl_ast_expr_get_ctx(
    __isl_keep isl_ast_expr *expr);
enum isl_ast_expr_type isl_ast_expr_get_type(
    __isl_keep isl_ast_expr *expr);
```

The type of an AST expression is one of `isl_ast_expr_op`, `isl_ast_expr_id` or `isl_ast_expr_int`. An `isl_ast_expr_op` represents the result of an operation. An `isl_ast_expr_id` represents an identifier. An `isl_ast_expr_int` represents an integer value.

Each type of expression has its own additional properties.

```
#include <isl/ast.h>
enum isl_ast_op_type isl_ast_expr_get_op_type(
    __isl_keep isl_ast_expr *expr);
int isl_ast_expr_get_op_n_arg(__isl_keep isl_ast_expr *expr);
__isl_give isl_ast_expr *isl_ast_expr_get_op_arg(
    __isl_keep isl_ast_expr *expr, int pos);
int isl_ast_node_foreach_ast_op_type(
    __isl_keep isl_ast_node *node,
    int (*fn)(enum isl_ast_op_type type, void *user),
    void *user);
```

`isl_ast_expr_get_op_type` returns the type of the operation performed. `isl_ast_expr_get_op_n_arg` returns the number of arguments. `isl_ast_expr_get_op_arg` returns the specified argument. `isl_ast_node_foreach_ast_op_type` calls `fn` for each distinct `isl_ast_op_type` that appears in `node`. The operation type is one of the following.

`isl_ast_op_and`

Logical *and* of two arguments. Both arguments can be evaluated.

`isl_ast_op_and_then`

Logical *and* of two arguments. The second argument can only be evaluated if the first evaluates to true.

`isl_ast_op_or`

Logical *or* of two arguments. Both arguments can be evaluated.

`isl_ast_op_or_else`

Logical *or* of two arguments. The second argument can only be evaluated if the first evaluates to false.

isl_ast_op_max

Maximum of two or more arguments.

isl_ast_op_min

Minimum of two or more arguments.

isl_ast_op_minus

Change sign.

isl_ast_op_add

Sum of two arguments.

isl_ast_op_sub

Difference of two arguments.

isl_ast_op_mul

Product of two arguments.

isl_ast_op_div

Exact division. That is, the result is known to be an integer.

isl_ast_op_fdiv_q

Result of integer division, rounded towards negative infinity.

isl_ast_op_pdiv_q

Result of integer division, where dividend is known to be non-negative.

isl_ast_op_pdiv_r

Remainder of integer division, where dividend is known to be non-negative.

isl_ast_op_cond

Conditional operator defined on three arguments. If the first argument evaluates to true, then the result is equal to the second argument. Otherwise, the result is equal to the third argument. The second and third argument may only be evaluated if the first argument evaluates to true and false, respectively. Corresponds to $a ? b : c$ in C.

isl_ast_op_select

Conditional operator defined on three arguments. If the first argument evaluates to true, then the result is equal to the second argument. Otherwise, the result is equal to the third argument. The second and third argument may be evaluated independently of the value of the first argument. Corresponds to $a * b + (1 - a) * c$ in C.

isl_ast_op_eq

Equality relation.

isl_ast_op_le

Less than or equal relation.

isl_ast_op_lt

Less than relation.

isl_ast_op_ge

Greater than or equal relation.

isl_ast_op_gt

Greater than relation.

isl_ast_op_call

A function call. The number of arguments of the `isl_ast_expr` is one more than the number of arguments in the function call, the first argument representing the function being called.

isl_ast_op_access

An array access. The number of arguments of the `isl_ast_expr` is one more than the number of index expressions in the array access, the first argument representing the array being accessed.

isl_ast_op_member

A member access. This operation has two arguments, a structure and the name of the member of the structure being accessed.

```
#include <isl/ast.h>
__isl_give isl_id *isl_ast_expr_get_id(
    __isl_keep isl_ast_expr *expr);
```

Return the identifier represented by the AST expression.

```
#include <isl/ast.h>
__isl_give isl_val *isl_ast_expr_get_val(
    __isl_keep isl_ast_expr *expr);
```

Return the integer represented by the AST expression.

Properties of ASTs

```
#include <isl/ast.h>
int isl_ast_expr_is_equal(__isl_keep isl_ast_expr *expr1,
    __isl_keep isl_ast_expr *expr2);
```

Check if two `isl_ast_exprs` are equal to each other.

Manipulating and printing the AST

AST nodes can be copied and freed using the following functions.

```
#include <isl/ast.h>
__isl_give isl_ast_node *isl_ast_node_copy(
    __isl_keep isl_ast_node *node);
__isl_null isl_ast_node *isl_ast_node_free(
    __isl_take isl_ast_node *node);
```

AST expressions can be copied and freed using the following functions.

```
#include <isl/ast.h>
__isl_give isl_ast_expr *isl_ast_expr_copy(
    __isl_keep isl_ast_expr *expr);
__isl_null isl_ast_expr *isl_ast_expr_free(
    __isl_take isl_ast_expr *expr);
```

New AST expressions can be created either directly or within the context of an `isl_ast_build`.

```
#include <isl/ast.h>
__isl_give isl_ast_expr *isl_ast_expr_from_val(
    __isl_take isl_val *v);
__isl_give isl_ast_expr *isl_ast_expr_from_id(
    __isl_take isl_id *id);
__isl_give isl_ast_expr *isl_ast_expr_neg(
    __isl_take isl_ast_expr *expr);
__isl_give isl_ast_expr *isl_ast_expr_address_of(
    __isl_take isl_ast_expr *expr);
__isl_give isl_ast_expr *isl_ast_expr_add(
    __isl_take isl_ast_expr *expr1,
    __isl_take isl_ast_expr *expr2);
__isl_give isl_ast_expr *isl_ast_expr_sub(
    __isl_take isl_ast_expr *expr1,
    __isl_take isl_ast_expr *expr2);
__isl_give isl_ast_expr *isl_ast_expr_mul(
    __isl_take isl_ast_expr *expr1,
    __isl_take isl_ast_expr *expr2);
__isl_give isl_ast_expr *isl_ast_expr_div(
    __isl_take isl_ast_expr *expr1,
    __isl_take isl_ast_expr *expr2);
__isl_give isl_ast_expr *isl_ast_expr_and(
    __isl_take isl_ast_expr *expr1,
    __isl_take isl_ast_expr *expr2);
__isl_give isl_ast_expr *isl_ast_expr_or(
    __isl_take isl_ast_expr *expr1,
```

```

__isl_take isl_ast_expr *expr2)
__isl_give isl_ast_expr *isl_ast_expr_eq(
    __isl_take isl_ast_expr *expr1,
    __isl_take isl_ast_expr *expr2);
__isl_give isl_ast_expr *isl_ast_expr_le(
    __isl_take isl_ast_expr *expr1,
    __isl_take isl_ast_expr *expr2);
__isl_give isl_ast_expr *isl_ast_expr_lt(
    __isl_take isl_ast_expr *expr1,
    __isl_take isl_ast_expr *expr2);
__isl_give isl_ast_expr *isl_ast_expr_ge(
    __isl_take isl_ast_expr *expr1,
    __isl_take isl_ast_expr *expr2);
__isl_give isl_ast_expr *isl_ast_expr_gt(
    __isl_take isl_ast_expr *expr1,
    __isl_take isl_ast_expr *expr2);
__isl_give isl_ast_expr *isl_ast_expr_access(
    __isl_take isl_ast_expr *array,
    __isl_take isl_ast_expr_list *indices);

```

The function `isl_ast_expr_address_of` can be applied to an `isl_ast_expr` of type `isl_ast_op_access` only. It is meant to represent the address of the `isl_ast_expr_access`.

```

#include <isl/ast_build.h>
__isl_give isl_ast_expr *isl_ast_build_expr_from_pw_aff(
    __isl_keep isl_ast_build *build,
    __isl_take isl_pw_aff *pa);
__isl_give isl_ast_expr *
isl_ast_build_access_from_pw_multi_aff(
    __isl_keep isl_ast_build *build,
    __isl_take isl_pw_multi_aff *pma);
__isl_give isl_ast_expr *
isl_ast_build_access_from_multi_pw_aff(
    __isl_keep isl_ast_build *build,
    __isl_take isl_multi_pw_aff *mpa);
__isl_give isl_ast_expr *
isl_ast_build_call_from_pw_multi_aff(
    __isl_keep isl_ast_build *build,
    __isl_take isl_pw_multi_aff *pma);
__isl_give isl_ast_expr *
isl_ast_build_call_from_multi_pw_aff(
    __isl_keep isl_ast_build *build,
    __isl_take isl_multi_pw_aff *mpa);

```

The domains of `pa`, `mpa` and `pma` should correspond to the schedule space of `build`. The tuple id of `mpa` or `pma` is used as the array being accessed or the function being called. If the accessed space is a nested relation, then it is taken to represent an access

of the member specified by the range of this nested relation of the structure specified by the domain of the nested relation.

The following functions can be used to modify an `isl_ast_expr`.

```
#include <isl/ast.h>
__isl_give isl_ast_expr *isl_ast_expr_set_op_arg(
    __isl_take isl_ast_expr *expr, int pos,
    __isl_take isl_ast_expr *arg);
```

Replace the argument of `expr` at position `pos` by `arg`.

```
#include <isl/ast.h>
__isl_give isl_ast_expr *isl_ast_expr_substitute_ids(
    __isl_take isl_ast_expr *expr,
    __isl_take isl_id_to_ast_expr *id2expr);
```

The function `isl_ast_expr_substitute_ids` replaces the subexpressions of `expr` of type `isl_ast_expr_id` by the corresponding expression in `id2expr`, if there is any.

User specified data can be attached to an `isl_ast_node` and obtained from the same `isl_ast_node` using the following functions.

```
#include <isl/ast.h>
__isl_give isl_ast_node *isl_ast_node_set_annotation(
    __isl_take isl_ast_node *node,
    __isl_take isl_id *annotation);
__isl_give isl_id *isl_ast_node_get_annotation(
    __isl_keep isl_ast_node *node);
```

Basic printing can be performed using the following functions.

```
#include <isl/ast.h>
__isl_give isl_printer *isl_printer_print_ast_expr(
    __isl_take isl_printer *p,
    __isl_keep isl_ast_expr *expr);
__isl_give isl_printer *isl_printer_print_ast_node(
    __isl_take isl_printer *p,
    __isl_keep isl_ast_node *node);
```

More advanced printing can be performed using the following functions.

```
#include <isl/ast.h>
__isl_give isl_printer *isl_ast_op_type_print_macro(
    enum isl_ast_op_type type,
    __isl_take isl_printer *p);
__isl_give isl_printer *isl_ast_node_print_macros(
    __isl_keep isl_ast_node *node,
    __isl_take isl_printer *p);
__isl_give isl_printer *isl_ast_node_print(
```

```

        __isl_keep isl_ast_node *node,
        __isl_take isl_printer *p,
        __isl_take isl_ast_print_options *options);
__isl_give isl_printer *isl_ast_node_for_print(
        __isl_keep isl_ast_node *node,
        __isl_take isl_printer *p,
        __isl_take isl_ast_print_options *options);
__isl_give isl_printer *isl_ast_node_if_print(
        __isl_keep isl_ast_node *node,
        __isl_take isl_printer *p,
        __isl_take isl_ast_print_options *options);

```

While printing an `isl_ast_node` in `ISL_FORMAT_C`, `isl` may print out an AST that makes use of macros such as `floord`, `min` and `max`. `isl_ast_op_type_print_macro` prints out the macro corresponding to a specific `isl_ast_op_type`. `isl_ast_node_print_macros` scans the `isl_ast_node` for expressions where these macros would be used and prints out the required macro definitions. Essentially, `isl_ast_node_print_macros` calls `isl_ast_node_foreach_ast_op_type` with `isl_ast_op_type_print_macro` as function argument. `isl_ast_node_print`, `isl_ast_node_for_print` and `isl_ast_node_if_print` print an `isl_ast_node` in `ISL_FORMAT_C`, but allow for some extra control through an `isl_ast_print_options` object. This object can be created using the following functions.

```

#include <isl/ast.h>
__isl_give isl_ast_print_options *
isl_ast_print_options_alloc(isl_ctx *ctx);
__isl_give isl_ast_print_options *
isl_ast_print_options_copy(
    __isl_keep isl_ast_print_options *options);
__isl_null isl_ast_print_options *
isl_ast_print_options_free(
    __isl_take isl_ast_print_options *options);

__isl_give isl_ast_print_options *
isl_ast_print_options_set_print_user(
    __isl_take isl_ast_print_options *options,
    __isl_give isl_printer *(*print_user)(
        __isl_take isl_printer *p,
        __isl_take isl_ast_print_options *options,
        __isl_keep isl_ast_node *node, void *user),
    void *user);
__isl_give isl_ast_print_options *
isl_ast_print_options_set_print_for(
    __isl_take isl_ast_print_options *options,
    __isl_give isl_printer *(*print_for)(
        __isl_take isl_printer *p,
        __isl_take isl_ast_print_options *options,

```

```

        __isl_keep isl_ast_node *node, void *user),
void *user);

```

The callback set by `isl_ast_print_options_set_print_user` is called whenever a node of type `isl_ast_node_user` needs to be printed. The callback set by `isl_ast_print_options_set_print_for` is called whenever a node of type `isl_ast_node_for` needs to be printed. Note that `isl_ast_node_for_print` will *not* call the callback set by `isl_ast_print_options_set_print_for` on the node on which `isl_ast_node_for_print` is called, but only on nested nodes of type `isl_ast_node_for`. It is therefore safe to call `isl_ast_node_for_print` from within the callback set by `isl_ast_print_options_set_print_for`.

The following option determines the type to be used for iterators while printing the AST.

```

int isl_options_set_ast_iterator_type(
    isl_ctx *ctx, const char *val);
const char *isl_options_get_ast_iterator_type(
    isl_ctx *ctx);

```

The AST printer only prints body nodes as blocks if these blocks cannot be safely omitted. For example, a `for` node with one body node will not be surrounded with braces in `ISL_FORMAT_C`. A block will always be printed by setting the following option.

```

int isl_options_set_ast_always_print_block(isl_ctx *ctx,
    int val);
int isl_options_get_ast_always_print_block(isl_ctx *ctx);

```

Options

```

#include <isl/ast_build.h>
int isl_options_set_ast_build_atomic_upper_bound(
    isl_ctx *ctx, int val);
int isl_options_get_ast_build_atomic_upper_bound(
    isl_ctx *ctx);
int isl_options_set_ast_build_prefer_pdiv(isl_ctx *ctx,
    int val);
int isl_options_get_ast_build_prefer_pdiv(isl_ctx *ctx);
int isl_options_set_ast_build_exploit_nested_bounds(
    isl_ctx *ctx, int val);
int isl_options_get_ast_build_exploit_nested_bounds(
    isl_ctx *ctx);
int isl_options_set_ast_build_group_coscheduled(
    isl_ctx *ctx, int val);
int isl_options_get_ast_build_group_coscheduled(
    isl_ctx *ctx);
int isl_options_set_ast_build_scale_strides(
    isl_ctx *ctx, int val);

```



```

int isl_options_get_ast_build_scale_strides(
    isl_ctx *ctx);
int isl_options_set_ast_build_allow_else(isl_ctx *ctx,
    int val);
int isl_options_get_ast_build_allow_else(isl_ctx *ctx);
int isl_options_set_ast_build_allow_or(isl_ctx *ctx,
    int val);
int isl_options_get_ast_build_allow_or(isl_ctx *ctx);

```

- `ast_build_atomic_upper_bound`

Generate loop upper bounds that consist of the current loop iterator, an operator and an expression not involving the iterator. If this option is not set, then the current loop iterator may appear several times in the upper bound. For example, when this option is turned off, AST generation for the schedule

$$[n] \rightarrow \{ A[i] \rightarrow [i] : 0 \leq i \leq 100, n \}$$

produces

```

for (int c0 = 0; c0 <= 100 && n >= c0; c0 += 1)
    A(c0);

```

When the option is turned on, the following AST is generated

```

for (int c0 = 0; c0 <= min(100, n); c0 += 1)
    A(c0);

```

- `ast_build_prefer_pdiv`

If this option is turned off, then the AST generation will produce ASTs that may only contain `isl_ast_op_fdiv_q` operators, but no `isl_ast_op_pdiv_q` or `isl_ast_op_pdiv_r` operators. If this option is turned on, then `isl` will try to convert some of the `isl_ast_op_fdiv_q` operators to (expressions containing) `isl_ast_op_pdiv_q` or `isl_ast_op_pdiv_r` operators.

- `ast_build_exploit_nested_bounds`

Simplify conditions based on bounds of nested for loops. In particular, remove conditions that are implied by the fact that one or more nested loops have at least one iteration, meaning that the upper bound is at least as large as the lower bound. For example, when this option is turned off, AST generation for the schedule

$$[N,M] \rightarrow \{ A[i,j] \rightarrow [i,j] : 0 \leq i \leq N \text{ and } 0 \leq j \leq M \}$$

produces

```

if (M >= 0)
  for (int c0 = 0; c0 <= N; c0 += 1)
    for (int c1 = 0; c1 <= M; c1 += 1)
      A(c0, c1);

```

When the option is turned on, the following AST is generated

```

for (int c0 = 0; c0 <= N; c0 += 1)
  for (int c1 = 0; c1 <= M; c1 += 1)
    A(c0, c1);

```

- `ast_build_group_coscheduled`

If two domain elements are assigned the same schedule point, then they may be executed in any order and they may even appear in different loops. If this options is set, then the AST generator will make sure that coscheduled domain elements do not appear in separate parts of the AST. This is useful in case of nested AST generation if the outer AST generation is given only part of a schedule and the inner AST generation should handle the domains that are coscheduled by this initial part of the schedule together. For example if an AST is generated for a schedule

```
{ A[i] -> [0]; B[i] -> [0] }
```

then the `isl_ast_build_set_create_leaf` callback described below may get called twice, once for each domain. Setting this option ensures that the callback is only called once on both domains together.

- `ast_build_separation_bounds`

This option specifies which bounds to use during separation. If this option is set to `ISL_AST_BUILD_SEPARATION_BOUNDS_IMPLICIT` then all (possibly implicit) bounds on the current dimension will be used during separation. If this option is set to `ISL_AST_BUILD_SEPARATION_BOUNDS_EXPLICIT` then only those bounds that are explicitly available will be used during separation.

- `ast_build_scale_strides`

This option specifies whether the AST generator is allowed to scale down iterators of strided loops.

- `ast_build_allow_else`

This option specifies whether the AST generator is allowed to construct if statements with else branches.

- `ast_build_allow_or`

This option specifies whether the AST generator is allowed to construct if conditions with disjunctions.

Fine-grained Control over AST Generation

Besides specifying the constraints on the parameters, an `isl_ast_build` object can be used to control various aspects of the AST generation process. The most prominent way of control is through “options”, which can be set using the following function.

```
#include <isl/ast_build.h>
__isl_give isl_ast_build *
isl_ast_build_set_options(
    __isl_take isl_ast_build *control,
    __isl_take isl_union_map *options);
```

The options are encoded in an `isl_union_map`. The domain of this union relation refers to the schedule domain, i.e., the range of the schedule passed to `isl_ast_build_ast_from_schedule`. In the case of nested AST generation (see §1.5.3), the domain of options should refer to the extra piece of the schedule. That is, it should be equal to the range of the wrapped relation in the range of the schedule. The range of the options can consist of elements in one or more spaces, the names of which determine the effect of the option. The values of the range typically also refer to the schedule dimension to which the option applies. In case of nested AST generation (see §1.5.3), these values refer to the position of the schedule dimension within the innermost AST generation. The constraints on the domain elements of the option should only refer to this dimension and earlier dimensions. We consider the following spaces.

separation_class

This space is a wrapped relation between two one dimensional spaces. The input space represents the schedule dimension to which the option applies and the output space represents the separation class. While constructing a loop corresponding to the specified schedule dimension(s), the AST generator will try to generate separate loops for domain elements that are assigned different classes. If only some of the elements are assigned a class, then those elements that are not assigned any class will be treated as belonging to a class that is separate from the explicitly assigned classes. The typical use case for this option is to separate full tiles from partial tiles. The other options, described below, are applied after the separation into classes.

As an example, consider the separation into full and partial tiles of a tiling of a triangular domain. Take, for example, the domain

$$\{ A[i,j] : 0 \leq i,j \text{ and } i + j \leq 100 \}$$

and a tiling into tiles of 10 by 10. The input to the AST generator is then the schedule

$$\{ A[i,j] \rightarrow [([i/10]), [j/10], i, j] : 0 \leq i,j \text{ and } i + j \leq 100 \}$$

Without any options, the following AST is generated

```

for (int c0 = 0; c0 <= 10; c0 += 1)
  for (int c1 = 0; c1 <= -c0 + 10; c1 += 1)
    for (int c2 = 10 * c0;
          c2 <= min(-10 * c1 + 100, 10 * c0 + 9);
          c2 += 1)
      for (int c3 = 10 * c1;
            c3 <= min(10 * c1 + 9, -c2 + 100);
            c3 += 1)
        A(c2, c3);

```

Separation into full and partial tiles can be obtained by assigning a class, say 0, to the full tiles. The full tiles are represented by those values of the first and second schedule dimensions for which there are values of the third and fourth dimensions to cover an entire tile. That is, we need to specify the following option

```

{ [a,b,c,d] -> separation_class[[0]->[0]] :
  exists b': 0 <= 10a,10b' and
             10a+9+10b'+9 <= 100;
  [a,b,c,d] -> separation_class[[1]->[0]] :
    0 <= 10a,10b and 10a+9+10b+9 <= 100 }

```

which simplifies to

```

{ [a, b, c, d] -> separation_class[[1] -> [0]] :
  a >= 0 and b >= 0 and b <= 8 - a;
  [a, b, c, d] -> separation_class[[0] -> [0]] :
    a >= 0 and a <= 8 }

```

With this option, the generated AST is as follows

```

{
  for (int c0 = 0; c0 <= 8; c0 += 1) {
    for (int c1 = 0; c1 <= -c0 + 8; c1 += 1)
      for (int c2 = 10 * c0;
            c2 <= 10 * c0 + 9; c2 += 1)
        for (int c3 = 10 * c1;
              c3 <= 10 * c1 + 9; c3 += 1)
          A(c2, c3);
    for (int c1 = -c0 + 9; c1 <= -c0 + 10; c1 += 1)
      for (int c2 = 10 * c0;
            c2 <= min(-10 * c1 + 100, 10 * c0 + 9);
            c2 += 1)

```

```

        for (int c3 = 10 * c1;
            c3 <= min(-c2 + 100, 10 * c1 + 9);
            c3 += 1)
            A(c2, c3);
    }
    for (int c0 = 9; c0 <= 10; c0 += 1)
        for (int c1 = 0; c1 <= -c0 + 10; c1 += 1)
            for (int c2 = 10 * c0;
                c2 <= min(-10 * c1 + 100, 10 * c0 + 9);
                c2 += 1)
                for (int c3 = 10 * c1;
                    c3 <= min(10 * c1 + 9, -c2 + 100);
                    c3 += 1)
                    A(c2, c3);
    }

```

separate

This is a single-dimensional space representing the schedule dimension(s) to which “separation” should be applied. Separation tries to split a loop into several pieces if this can avoid the generation of guards inside the loop. See also the `atomic` option.

atomic

This is a single-dimensional space representing the schedule dimension(s) for which the domains should be considered “atomic”. That is, the AST generator will make sure that any given domain space will only appear in a single loop at the specified level.

Consider the following schedule

```

{ a[i] -> [i] : 0 <= i < 10;
  b[i] -> [i+1] : 0 <= i < 10 }

```

If the following option is specified

```

{ [i] -> separate[x] }

```

then the following AST will be generated

```

{
  a(0);
  for (int c0 = 1; c0 <= 9; c0 += 1) {
    a(c0);
    b(c0 - 1);
  }
  b(9);
}

```

If, on the other hand, the following option is specified

```
{ [i] -> atomic[x] }
```

then the following AST will be generated

```
for (int c0 = 0; c0 <= 10; c0 += 1) {  
    if (c0 <= 9)  
        a(c0);  
    if (c0 >= 1)  
        b(c0 - 1);  
}
```

If neither `atomic` nor `separate` is specified, then the AST generator may produce either of these two results or some intermediate form.

unroll

This is a single-dimensional space representing the schedule dimension(s) that should be *completely* unrolled. To obtain a partial unrolling, the user should apply an additional strip-mining to the schedule and fully unroll the inner loop.

Additional control is available through the following functions.

```
#include <isl/ast_build.h>  
__isl_give isl_ast_build *  
isl_ast_build_set_iterators(  
    __isl_take isl_ast_build *control,  
    __isl_take isl_id_list *iterators);
```

The function `isl_ast_build_set_iterators` allows the user to specify a list of iterator `isl_ids` to be used as iterators. If the input schedule is injective, then the number of elements in this list should be as large as the dimension of the schedule space, but no direct correspondence should be assumed between dimensions and elements. If the input schedule is not injective, then an additional number of `isl_ids` equal to the largest dimension of the input domains may be required. If the number of provided `isl_ids` is insufficient, then additional names are automatically generated.

```
#include <isl/ast_build.h>  
__isl_give isl_ast_build *  
isl_ast_build_set_create_leaf(  
    __isl_take isl_ast_build *control,  
    __isl_give isl_ast_node *(*fn)(  
        __isl_take isl_ast_build *build,  
        void *user), void *user);
```

The `isl_ast_build_set_create_leaf` function allows for the specification of a callback that should be called whenever the AST generator arrives at an element of the schedule domain. The callback should return an AST node that should be inserted at the corresponding position of the AST. The default action (when the callback is not set) is to continue generating parts of the AST to scan all the domain elements associated to the schedule domain element and to insert user nodes, “calling” the domain element, for each of them. The build argument contains the current state of the `isl_ast_build`. To ease nested AST generation (see §1.5.3), all control information that is specific to the current AST generation such as the options and the callbacks has been removed from this `isl_ast_build`. The callback would typically return the result of a nested AST generation or a user defined node created using the following function.

```
#include <isl/ast.h>
__isl_give isl_ast_node *isl_ast_node_alloc_user(
    __isl_take isl_ast_expr *expr);

#include <isl/ast_build.h>
__isl_give isl_ast_build *
isl_ast_build_set_at_each_domain(
    __isl_take isl_ast_build *build,
    __isl_give isl_ast_node *(*fn)(
        __isl_take isl_ast_node *node,
        __isl_keep isl_ast_build *build,
        void *user), void *user);
__isl_give isl_ast_build *
isl_ast_build_set_before_each_for(
    __isl_take isl_ast_build *build,
    __isl_give isl_id *(*fn)(
        __isl_keep isl_ast_build *build,
        void *user), void *user);
__isl_give isl_ast_build *
isl_ast_build_set_after_each_for(
    __isl_take isl_ast_build *build,
    __isl_give isl_ast_node *(*fn)(
        __isl_take isl_ast_node *node,
        __isl_keep isl_ast_build *build,
        void *user), void *user);
```

The callback set by `isl_ast_build_set_at_each_domain` will be called for each domain AST node. The callbacks set by `isl_ast_build_set_before_each_for` and `isl_ast_build_set_after_each_for` will be called for each for AST node. The first will be called in depth-first pre-order, while the second will be called in depth-first post-order. Since `isl_ast_build_set_before_each_for` is called before the for node is actually constructed, it is only passed an `isl_ast_build`. The returned `isl_id` will be added as an annotation (using `isl_ast_node_set_annotation`) to the constructed for node. In particular, if the user has also specified an `after_each_for`

callback, then the annotation can be retrieved from the node passed to that callback using `isl_ast_node_get_annotation`. All callbacks should return NULL on failure. The given `isl_ast_build` can be used to create new `isl_ast_expr` objects using `isl_ast_build_expr_from_pw_aff` or `isl_ast_build_call_from_pw_multi_aff`.

Nested AST Generation

`isl` allows the user to create an AST within the context of another AST. These nested ASTs are created using the same `isl_ast_build_ast_from_schedule` function that is used to create the outer AST. The build argument should be an `isl_ast_build` passed to a callback set by `isl_ast_build_set_create_leaf`. The space of the range of the `schedule` argument should refer to this build. In particular, the space should be a wrapped relation and the domain of this wrapped relation should be the same as that of the range of the schedule returned by `isl_ast_build_get_schedule` below. In practice, the new schedule is typically created by calling `isl_union_map_range_product` on the old schedule and some extra piece of the schedule. The space of the schedule domain is also available from the `isl_ast_build`.

```
#include <isl/ast_build.h>
__isl_give isl_union_map *isl_ast_build_get_schedule(
    __isl_keep isl_ast_build *build);
__isl_give isl_space *isl_ast_build_get_schedule_space(
    __isl_keep isl_ast_build *build);
__isl_give isl_ast_build *isl_ast_build_restrict(
    __isl_take isl_ast_build *build,
    __isl_take isl_set *set);
```

The `isl_ast_build_get_schedule` function returns a (partial) schedule for the domain elements for which part of the AST still needs to be generated in the current build. In particular, the domain elements are mapped to those iterations of the loops enclosing the current point of the AST generation inside which the domain elements are executed. No direct correspondence between the input schedule and this schedule should be assumed. The space obtained from `isl_ast_build_get_schedule_space` can be used to create a set for `isl_ast_build_restrict` to intersect with the current build. In particular, the set passed to `isl_ast_build_restrict` can have additional parameters. The ids of the set dimensions in the space returned by `isl_ast_build_get_schedule_space` correspond to the iterators of the already generated loops. The user should not rely on the ids of the output dimensions of the relations in the union relation returned by `isl_ast_build_get_schedule` having any particular value.

1.6 Applications

Although `isl` is mainly meant to be used as a library, it also contains some basic applications that use some of the functionality of `isl`. The input may be specified in either the `isl` format or the PolyLib format.

1.6.1 `isl_polyhedron_sample`

`isl_polyhedron_sample` takes a polyhedron as input and prints an integer element of the polyhedron, if there is any. The first column in the output is the denominator and is always equal to 1. If the polyhedron contains no integer points, then a vector of length zero is printed.

1.6.2 `isl_pip`

`isl_pip` takes the same input as the `example` program from the `piplib` distribution, i.e., a set of constraints on the parameters, a line containing only -1 and finally a set of constraints on a parametric polyhedron. The coefficients of the parameters appear in the last columns (but before the final constant column). The output is the lexicographic minimum of the parametric polyhedron. As `isl` currently does not have its own output format, the output is just a dump of the internal state.

1.6.3 `isl_polyhedron_minimize`

`isl_polyhedron_minimize` computes the minimum of some linear or affine objective function over the integer points in a polyhedron. If an affine objective function is given, then the constant should appear in the last column.

1.6.4 `isl_polytope_scan`

Given a polytope, `isl_polytope_scan` prints all integer points in the polytope.

1.6.5 `isl_codegen`

Given a schedule, a context set and an options relation, `isl_codegen` prints out an AST that scans the domain elements of the schedule in the order of their image(s) taking into account the constraints in the context set.

Chapter 2

Implementation Details

2.1 Sets and Relations

Definition 2.1.1 (Polyhedral Set) A polyhedral set S is a finite union of basic sets $S = \bigcup_i S_i$, each of which can be represented using affine constraints

$$S_i : \mathbb{Z}^n \rightarrow 2^{\mathbb{Z}^d} : \mathbf{s} \mapsto S_i(\mathbf{s}) = \{ \mathbf{x} \in \mathbb{Z}^d \mid \exists \mathbf{z} \in \mathbb{Z}^e : A\mathbf{x} + B\mathbf{s} + D\mathbf{z} + \mathbf{c} \geq \mathbf{0} \},$$

with $A \in \mathbb{Z}^{m \times d}$, $B \in \mathbb{Z}^{m \times n}$, $D \in \mathbb{Z}^{m \times e}$ and $\mathbf{c} \in \mathbb{Z}^m$.

Definition 2.1.2 (Parameter Domain of a Set) Let $S \in \mathbb{Z}^n \rightarrow 2^{\mathbb{Z}^d}$ be a set. The parameter domain of S is the set

$$\text{pdom } S := \{ \mathbf{s} \in \mathbb{Z}^n \mid S(\mathbf{s}) \neq \emptyset \}.$$

Definition 2.1.3 (Polyhedral Relation) A polyhedral relation R is a finite union of basic relations $R = \bigcup_i R_i$ of type $\mathbb{Z}^n \rightarrow 2^{\mathbb{Z}^{d_1+d_2}}$, each of which can be represented using affine constraints

$$R_i = \mathbf{s} \mapsto R_i(\mathbf{s}) = \{ \mathbf{x}_1 \rightarrow \mathbf{x}_2 \in \mathbb{Z}^{d_1} \times \mathbb{Z}^{d_2} \mid \exists \mathbf{z} \in \mathbb{Z}^e : A_1\mathbf{x}_1 + A_2\mathbf{x}_2 + B\mathbf{s} + D\mathbf{z} + \mathbf{c} \geq \mathbf{0} \},$$

with $A_i \in \mathbb{Z}^{m \times d_i}$, $B \in \mathbb{Z}^{m \times n}$, $D \in \mathbb{Z}^{m \times e}$ and $\mathbf{c} \in \mathbb{Z}^m$.

Definition 2.1.4 (Parameter Domain of a Relation) Let $R \in \mathbb{Z}^n \rightarrow 2^{\mathbb{Z}^{d+d}}$ be a relation. The parameter domain of R is the set

$$\text{pdom } R := \{ \mathbf{s} \in \mathbb{Z}^n \mid R(\mathbf{s}) \neq \emptyset \}.$$

Definition 2.1.5 (Domain of a Relation) Let $R \in \mathbb{Z}^n \rightarrow 2^{\mathbb{Z}^{d+d}}$ be a relation. The domain of R is the polyhedral set

$$\text{dom } R := \mathbf{s} \mapsto \{ \mathbf{x}_1 \in \mathbb{Z}^{d_1} \mid \exists \mathbf{x}_2 \in \mathbb{Z}^{d_2} : (\mathbf{x}_1, \mathbf{x}_2) \in R(\mathbf{s}) \}.$$

Definition 2.1.6 (Range of a Relation) Let $R \in \mathbb{Z}^n \rightarrow 2^{\mathbb{Z}^{d+d}}$ be a relation. The range of R is the polyhedral set

$$\text{ran } R := \mathbf{s} \mapsto \{ \mathbf{x}_2 \in \mathbb{Z}^{d_2} \mid \exists \mathbf{x}_1 \in \mathbb{Z}^{d_1} : (\mathbf{x}_1, \mathbf{x}_2) \in R(\mathbf{s}) \}.$$

Definition 2.1.7 (Composition of Relations) Let $R \in \mathbb{Z}^n \rightarrow 2^{\mathbb{Z}^{d_1+d_2}}$ and $S \in \mathbb{Z}^n \rightarrow 2^{\mathbb{Z}^{d_2+d_3}}$ be two relations, then the composition of R and S is defined as

$$S \circ R := \mathbf{s} \mapsto \{ \mathbf{x}_1 \rightarrow \mathbf{x}_3 \in \mathbb{Z}^{d_1} \times \mathbb{Z}^{d_3} \mid \exists \mathbf{x}_2 \in \mathbb{Z}^{d_2} : \mathbf{x}_1 \rightarrow \mathbf{x}_2 \in R(\mathbf{s}) \wedge \mathbf{x}_2 \rightarrow \mathbf{x}_3 \in S(\mathbf{s}) \}.$$

Definition 2.1.8 (Difference Set of a Relation) Let $R \in \mathbb{Z}^n \rightarrow 2^{\mathbb{Z}^{d+d}}$ be a relation. The difference set (ΔR) of R is the set of differences between image elements and the corresponding domain elements,

$$\Delta R := \mathbf{s} \mapsto \{ \boldsymbol{\delta} \in \mathbb{Z}^d \mid \exists \mathbf{x} \rightarrow \mathbf{y} \in R : \boldsymbol{\delta} = \mathbf{y} - \mathbf{x} \}$$

2.2 Simple Hull

It is sometimes useful to have a single basic set or basic relation that contains a given set or relation. For rational sets, the obvious choice would be to compute the (rational) convex hull. For integer sets, the obvious choice would be the integer hull. However, isl currently does not support an integer hull operation and even if it did, it would be fairly expensive to compute. The convex hull operation is supported, but it is also fairly expensive to compute given only an implicit representation.

Usually, it is not required to compute the exact integer hull, and an overapproximation of this hull is sufficient. The “simple hull” of a set is such an overapproximation and it is defined as the (inclusion-wise) smallest basic set that is described by constraints that are translates of the constraints in the input set. This means that the simple hull is relatively cheap to compute and that the number of constraints in the simple hull is no larger than the number of constraints in the input.

Definition 2.2.1 (Simple Hull of a Set) The simple hull of a set $S = \bigcup_{1 \leq i \leq v} S_i$, with

$$S : \mathbb{Z}^n \rightarrow 2^{\mathbb{Z}^d} : \mathbf{s} \mapsto S(\mathbf{s}) = \left\{ \mathbf{x} \in \mathbb{Z}^d \mid \exists \mathbf{z} \in \mathbb{Z}^e : \bigvee_{1 \leq i \leq v} A_i \mathbf{x} + B_i \mathbf{s} + D_i \mathbf{z} + \mathbf{c}_i \geq \mathbf{0} \right\}$$

is the set

$$H : \mathbb{Z}^n \rightarrow 2^{\mathbb{Z}^d} : \mathbf{s} \mapsto S(\mathbf{s}) = \left\{ \mathbf{x} \in \mathbb{Z}^d \mid \exists \mathbf{z} \in \mathbb{Z}^e : \bigwedge_{1 \leq i \leq v} A_i \mathbf{x} + B_i \mathbf{s} + D_i \mathbf{z} + \mathbf{c}_i + \mathbf{K}_i \geq \mathbf{0} \right\},$$

with \mathbf{K}_i the (component-wise) smallest non-negative integer vectors such that $S \subseteq H$.

The \mathbf{K}_i can be obtained by solving a number of LP problems, one for each element of each \mathbf{K}_i . If any LP problem is unbounded, then the corresponding constraint is dropped.

2.3 Parametric Integer Programming

2.3.1 Introduction

Parametric integer programming (Feautrier 1988) is used to solve many problems within the context of the polyhedral model. Here, we are mainly interested in dependence analysis (Feautrier 1991) and in computing a unique representation for existentially quantified variables. The latter operation has been used for counting elements in sets involving such variables (Boulet and Redon 1998; Verdoolaege et al. 2005) and lies at the core of the internal representation of `isl`.

Parametric integer programming was first implemented in `PipLib`. An alternative method for parametric integer programming was later implemented in `barvinok` (Verdoolaege 2006). This method is not based on Feautrier’s algorithm, but on rational generating functions (Barvinok and Woods 2003) and was inspired by the “digging” technique of De Loera et al. (2004) for solving non-parametric integer programming problems.

In the following sections, we briefly recall the dual simplex method combined with Gomory cuts and describe some extensions and optimizations. The main algorithm is applied to a matrix data structure known as a tableau. In case of parametric problems, there are two tableaus, one for the main problem and one for the constraints on the parameters, known as the context tableau. The handling of the context tableau is described in Section 2.3.7.

2.3.2 The Dual Simplex Method

Tableaus can be represented in several slightly different ways. In `isl`, the dual simplex method uses the same representation as that used by its incremental LP solver based on the *primal* simplex method. The implementation of this LP solver is based on that of `Simplify` (Detlefs et al. 2005), which, in turn, was derived from the work of Nelson (1980). In the original (Nelson 1980), the tableau was implemented as a sparse matrix, but neither `Simplify` nor the current implementation of `isl` does so.

Given some affine constraints on the variables, $A\mathbf{x} + \mathbf{b} \geq \mathbf{0}$, the tableau represents the relationship between the variables \mathbf{x} and non-negative variables $\mathbf{y} = A\mathbf{x} + \mathbf{b}$ corresponding to the constraints. The initial tableau contains $(\mathbf{b} \ A)$ and expresses the constraints \mathbf{y} in the rows in terms of the variables \mathbf{x} in the columns. The main operation defined on a tableau exchanges a column and a row variable and is called a pivot. During this process, some coefficients may become rational. As in the `PipLib` implementation, `isl` maintains a shared denominator per row. The sample value of a tableau is one where each column variable is assigned zero and each row variable is assigned the constant term of the row. This sample value represents a valid solution if each constraint variable is assigned a non-negative value, i.e., if the constant terms of rows corresponding to constraints are all non-negative.

The dual simplex method starts from an initial sample value that may be invalid, but that is known to be (lexicographically) no greater than any solution, and gradually increments this sample value through pivoting until a valid solution is obtained. In particular, each pivot exchanges a row variable $r = -n + \sum_i a_i c_i$ with negative sample

value $-n$ with a column variable c_j such that $a_j > 0$. Since $c_j = (n + r - \sum_{i \neq j} a_i c_i)/a_j$, the new row variable will have a positive sample value n . If no such column can be found, then the problem is infeasible. By always choosing the column that leads to the (lexicographically) smallest increment in the variables \mathbf{x} , the first solution found is guaranteed to be the (lexicographically) minimal solution (Feautrier 1988). In order to be able to determine the smallest increment, the tableau is (implicitly) extended with extra rows defining the original variables in terms of the column variables. If we assume that all variables are non-negative, then we know that the zero vector is no greater than the minimal solution and then the initial extended tableau looks as follows.

$$\begin{array}{c} \mathbf{x} \\ \mathbf{r} \end{array} \left(\begin{array}{cc} 1 & \mathbf{c} \\ \mathbf{0} & I \\ \mathbf{b} & A \end{array} \right)$$

Each column in this extended tableau is lexicographically positive and will remain so because of the column choice explained above. It is then clear that the value of \mathbf{x} will increase in each step. Note that there is no need to store the extra rows explicitly. If a given x_i is a column variable, then the corresponding row is the unit vector e_i . If, on the other hand, it is a row variable, then the row already appears somewhere else in the tableau.

In case of parametric problems, the sign of the constant term may depend on the parameters. Each time the constant term of a constraint row changes, we therefore need to check whether the new term can attain negative and/or positive values over the current set of possible parameter values, i.e., the context. If all these terms can only attain non-negative values, the current state of the tableau represents a solution. If one of the terms can only attain non-positive values and is not identically zero, the corresponding row can be pivoted. Otherwise, we pick one of the terms that can attain both positive and negative values and split the context into a part where it only attains non-negative values and a part where it only attains negative values.

2.3.3 Gomory Cuts

The solution found by the dual simplex method may have non-integral coordinates. If so, some rational solutions (including the current sample value), can be cut off by applying a (parametric) Gomory cut. Let $r = b(\mathbf{p}) + \langle \mathbf{a}, \mathbf{c} \rangle$ be the row corresponding to the first non-integral coordinate of \mathbf{x} , with $b(\mathbf{p})$ the constant term, an affine expression in the parameters \mathbf{p} , i.e., $b(\mathbf{p}) = \langle \mathbf{f}, \mathbf{p} \rangle + g$. Note that only row variables can attain non-integral values as the sample value of the column variables is zero. Consider the expression $b(\mathbf{p}) - \lceil b(\mathbf{p}) \rceil + \langle \{\mathbf{a}\}, \mathbf{c} \rangle$, with $\lceil \cdot \rceil$ the ceiling function and $\{\cdot\}$ the fractional part. This expression is negative at the sample value since $\mathbf{c} = \mathbf{0}$ and $r = b(\mathbf{p})$ is fractional, i.e., $\lceil b(\mathbf{p}) \rceil > b(\mathbf{p})$. On the other hand, for each integral value of r and $\mathbf{c} \geq 0$, the expression is non-negative because $b(\mathbf{p}) - \lceil b(\mathbf{p}) \rceil > -1$. Imposing this expression to be non-negative therefore does not invalidate any integral solutions, while it does cut away the current fractional sample value. To be able to formulate this constraint, a new variable $q = \lfloor -b(\mathbf{p}) \rfloor = -\lceil b(\mathbf{p}) \rceil$ is added to the context. This integral variable

is uniquely defined by the constraints $0 \leq -db(\mathbf{p}) - dq \leq d - 1$, with d the common denominator of \mathbf{f} and g . In practice, the variable $q' = \lfloor \langle \{-f\}, \mathbf{p} \rangle + \{-g\} \rfloor$ is used instead and the coefficients of the new constraint are adjusted accordingly. The sign of the constant term of this new constraint need not be determined as it is non-positive by construction. When several of these extra context variables are added, it is important to avoid adding duplicates. Recent versions of PipLib also check for such duplicates.

2.3.4 Negative Unknowns and Maximization

There are two places in the above algorithm where the unknowns \mathbf{x} are assumed to be non-negative: the initial tableau starts from sample value $\mathbf{x} = \mathbf{0}$ and \mathbf{c} is assumed to be non-negative during the construction of Gomory cuts. To deal with negative unknowns, Feautrier (1991, Appendix A.2) proposed to use a “big parameter”, say M , that is taken to be an arbitrarily large positive number. Instead of looking for the lexicographically minimal value of \mathbf{x} , we search instead for the lexicographically minimal value of $\mathbf{x}' = \mathbf{M} + \mathbf{x}$. The sample value $\mathbf{x}' = \mathbf{0}$ of the initial tableau then corresponds to $\mathbf{x} = -\mathbf{M}$, which is clearly not greater than any potential solution. The sign of the constant term of a row is determined lexicographically, with the coefficient of M considered first. That is, if the coefficient of M is not zero, then its sign is the sign of the entire term. Otherwise, the sign is determined by the remaining affine expression in the parameters. If the original problem has a bounded optimum, then the final sample value will be of the form $\mathbf{M} + \mathbf{v}$ and the optimal value of the original problem is then \mathbf{v} . Maximization problems can be handled in a similar way by computing the minimum of $\mathbf{M} - \mathbf{x}$.

When the optimum is unbounded, the optimal value computed for the original problem will involve the big parameter. In the original implementation of PipLib, the big parameter could even appear in some of the extra variables \mathbf{q} created during the application of a Gomory cut. The final result could then contain implicit conditions on the big parameter through conditions on such \mathbf{q} variables. This problem was resolved in later versions of PipLib by taking M to be divisible by any positive number. The big parameter can then never appear in any \mathbf{q} because $\{\alpha M\} = 0$. It should be noted, though, that an unbounded problem usually (but not always) indicates an incorrect formulation of the problem.

The original version of PipLib required the user to “manually” add a big parameter, perform the reformulation and interpret the result (Feautrier et al. 2002). Recent versions allow the user to simply specify that the unknowns may be negative or that the maximum should be computed and then these transformations are performed internally. Although there are some application, e.g., that of Feautrier (1992), where it is useful to have explicit control over the big parameter, negative unknowns and maximization are by far the most common applications of the big parameter and we believe that the user should not be bothered with such implementation issues. The current version of isl therefore does not provide any interface for specifying big parameters. Instead, the user can specify whether a maximum needs to be computed and no assumptions are made on the sign of the unknowns. Instead, the sign of the unknowns is checked internally and a big parameter is automatically introduced when needed. For compatibility with PipLib, the `isl_pip` tool does explicitly add non-negativity constraints on the unknowns unless the `Urs_unknowns` option is specified. Currently, there is also no

way in `isl` of expressing a big parameter in the output. Even though `isl` makes the same divisibility assumption on the big parameter as recent versions of `PipLib`, it will therefore eventually produce an error if the problem turns out to be unbounded.

2.3.5 Preprocessing

In this section, we describe some transformations that are or can be applied in advance to reduce the running time of the actual dual simplex method with Gomory cuts.

Feasibility Check and Detection of Equalities

Experience with the original `PipLib` has shown that Gomory cuts do not perform very well on problems that are (non-obviously) empty, i.e., problems with rational solutions, but no integer solutions. In `isl`, we therefore first perform a feasibility check on the original problem considered as a non-parametric problem over the combined space of unknowns and parameters. In fact, we do not simply check the feasibility, but we also check for implicit equalities among the integer points by computing the integer affine hull. The algorithm used is the same as that described in Section 2.3.7 below. Computing the affine hull is fairly expensive, but it can bring huge benefits if any equalities can be found or if the problem turns out to be empty.

Constraint Simplification

If the coefficients of the unknown and parameters in a constraint have a common factor, then this factor should be removed, possibly rounding down the constant term. For example, the constraint $2x - 5 \geq 0$ should be simplified to $x - 3 \geq 0$. `isl` performs such simplifications on all sets and relations. Recent versions of `PipLib` also perform this simplification on the input.

Exploiting Equalities

If there are any (explicit) equalities in the input description, `PipLib` converts each into a pair of inequalities. It is also possible to write r equalities as $r + 1$ inequalities (Feautrier et al. 2002), but it is even better to *exploit* the equalities to reduce the dimensionality of the problem. Given an equality involving at least one unknown, we pivot the row corresponding to the equality with the column corresponding to the last unknown with non-zero coefficient. The new column variable can then be removed completely because it is identically zero, thereby reducing the dimensionality of the problem by one. The last unknown is chosen to ensure that the columns of the initial tableau remain lexicographically positive. In particular, if the equality is of the form $b + \sum_{i \leq j} a_i x_i = 0$ with $a_j \neq 0$, then the (implicit) top rows of the initial tableau are

changed as follows

$$j \begin{pmatrix} & j \\ 0 & I_1 \\ 0 & & 1 \\ 0 & & & I_2 \end{pmatrix} \rightsquigarrow j \begin{pmatrix} 0 & I_1 \\ -b/a_j & -a_i/a_j \\ 0 & & I_2 \end{pmatrix}$$

Currently, `isl` also eliminates equalities involving only parameters in a similar way, provided at least one of the coefficients is equal to one. The application of parameter compression (see below) would obviate the need for removing parametric equalities.

Offline Symmetry Detection

Some problems, notably those of Bygde (2010), have a collection of constraints, say $b_i(\mathbf{p}) + \langle \mathbf{a}, \mathbf{x} \rangle \geq 0$, that only differ in their (parametric) constant terms. These constant terms will be non-negative on different parts of the context and this context may have to be split for each of the constraints. In the worst case, the basic algorithm may have to consider all possible orderings of the constant terms. Instead, `isl` introduces a new parameter, say u , and replaces the collection of constraints by the single constraint $u + \langle \mathbf{a}, \mathbf{x} \rangle \geq 0$ along with context constraints $u \leq b_i(\mathbf{p})$. Any solution to the new system is also a solution to the original system since $\langle \mathbf{a}, \mathbf{x} \rangle \geq -u \geq -b_i(\mathbf{p})$. Conversely, $m = \min_i b_i(\mathbf{p})$ satisfies the constraints on u and therefore extends a solution to the new system. It can also be plugged into a new solution. See Section 2.3.6 for how this substitution is currently performed in `isl`. The method described in this section can only detect symmetries that are explicitly available in the input. See Section 2.3.9 for the detection and exploitation of symmetries that appear during the course of the dual simplex method.

Parameter Compression

It may in some cases be apparent from the equalities in the problem description that there can only be a solution for a sublattice of the parameters. In such cases “parameter compression” (Meister 2004; Meister and Verdoolaege 2008) can be used to replace the parameters by alternative “dense” parameters. For example, if there is a constraint $2x = n$, then the system will only have solutions for even values of n and n can be replaced by $2n'$. Similarly, the parameters n and m in a system with the constraint $2n = 3m$ can be replaced by a single parameter n' with $n = 3n'$ and $m = 2n'$. It is also possible to perform a similar compression on the unknowns, but it would be more complicated as the compression would have to preserve the lexicographical order. Moreover, due to our handling of equalities described above there should be no need for such variable compression. Although parameter compression has been implemented in `isl`, it is currently not yet used during parametric integer programming.

2.3.6 Postprocessing

The output of `PipLib` is a quast (quasi-affine selection tree). Each internal node in this tree corresponds to a split of the context based on a parametric constant term in the main

tableau with indeterminate sign. Each of these nodes may introduce extra variables in the context corresponding to integer divisions. Each leaf of the tree prescribes the solution in that part of the context that satisfies all the conditions on the path leading to the leaf. Such a quast is a very economical way of representing the solution, but it would not be suitable as the (only) internal representation of sets and relations in `isl`. Instead, `isl` represents the constraints of a set or relation in disjunctive normal form. The result of a parametric integer programming problem is then also converted to this internal representation. Unfortunately, the conversion to disjunctive normal form can lead to an explosion of the size of the representation. In some cases, this overhead would have to be paid anyway in subsequent operations, but in other cases, especially for outside users that just want to solve parametric integer programming problems, we would like to avoid this overhead in future. That is, we are planning on introducing quasts or a related representation as one of several possible internal representations and on allowing the output of `isl_pip` to optionally be printed as a quast.

Currently, `isl` also does not have an internal representation for expressions such as $\min_i b_i(\mathbf{p})$ from the offline symmetry detection of Section 2.3.5. Assume that one of these expressions has n bounds $b_i(\mathbf{p})$. If the expression does not appear in the affine expression describing the solution, but only in the constraints, and if moreover, the expression only appears with a positive coefficient, i.e., $\min_i b_i(\mathbf{p}) \geq f_j(\mathbf{p})$, then each of these constraints can simply be reduplicated n times, once for each of the bounds. Otherwise, a conversion to disjunctive normal form leads to n cases, each described as $u = b_i(\mathbf{p})$ with constraints $b_i(\mathbf{p}) \leq b_j(\mathbf{p})$ for $j > i$ and $b_i(\mathbf{p}) < b_j(\mathbf{p})$ for $j < i$. Note that even though this conversion leads to a size increase by a factor of n , not detecting the symmetry could lead to an increase by a factor of $n!$ if all possible orderings end up being considered.

2.3.7 Context Tableau

The main operation that a context tableau needs to provide is a test on the sign of an affine expression over the elements of the context. This sign can be determined by solving two integer linear feasibility problems, one with a constraint added to the context that enforces the expression to be non-negative and one where the expression is negative. As already mentioned by Feautrier (1988), any integer linear feasibility solver could be used, but the `PipLib` implementation uses a recursive call to the dual simplex with Gomory cuts algorithm to determine the feasibility of a context. In `isl`, two ways of handling the context have been implemented, one that performs the recursive call and one, used by default, that uses generalized basis reduction. We start with some optimizations that are shared between the two implementations and then discuss additional details of each of them.

Maintaining Witnesses

A common feature of both integer linear feasibility solvers is that they will not only say whether a set is empty or not, but if the set is non-empty, they will also provide a *witness* for this result, i.e., a point that belongs to the set. By maintaining a list of such witnesses, we can avoid many feasibility tests during the determination of the signs

of affine expressions. In particular, if the expression evaluates to a positive number on some of these points and to a negative number on some others, then no feasibility test needs to be performed. If all the evaluations are non-negative, we only need to check for the possibility of a negative value and similarly in case of all non-positive evaluations. Finally, in the rare case that all points evaluate to zero or at the start, when no points have been collected yet, one or two feasibility tests need to be performed depending on the result of the first test.

When a new constraint is added to the context, the points that violate the constraint are temporarily removed. They are reconsidered when we backtrack over the addition of the constraint, as they will satisfy the negation of the constraint. It is only when we backtrack over the addition of the points that they are finally removed completely. When an extra integer division is added to the context, the new coordinates of the witnesses can easily be computed by evaluating the integer division. The idea of keeping track of witnesses was first used in `barvinok`.

Choice of Constant Term on which to Split

Recall that if there are no rows with a non-positive constant term, but there are rows with an indeterminate sign, then the context needs to be split along the constant term of one of these rows. If there is more than one such row, then we need to choose which row to split on first. `PipLib` uses a heuristic based on the (absolute) sizes of the coefficients. In particular, it takes the largest coefficient of each row and then selects the row where this largest coefficient is smaller than those of the other rows.

In `isl`, we take that row for which non-negativity of its constant term implies non-negativity of as many of the constant terms of the other rows as possible. The intuition behind this heuristic is that on the positive side, we will have fewer negative and indeterminate signs, while on the negative side, we need to perform a pivot, which may affect any number of rows meaning that the effect on the signs is difficult to predict. This heuristic is of course much more expensive to evaluate than the heuristic used by `PipLib`. More extensive tests are needed to evaluate whether the heuristic is worthwhile.

Dual Simplex + Gomory Cuts

When a new constraint is added to the context, the first steps of the dual simplex method applied to this new context will be the same or at least very similar to those taken on the original context, i.e., before the constraint was added. In `isl`, we therefore apply the dual simplex method incrementally on the context and backtrack to a previous state when a constraint is removed again. An initial implementation that was never made public would also keep the Gomory cuts, but the current implementation backtracks to before the point where Gomory cuts are added before adding an extra constraint to the context. Keeping the Gomory cuts has the advantage that the sample value is always an integer point and that this point may also satisfy the new constraint. However, due to the technique of maintaining witnesses explained above, we would not perform a feasibility test in such cases and then the previously added cuts may be redundant, possibly resulting in an accumulation of a large number of cuts.

If the parameters may be negative, then the same big parameter trick used in the main tableau is applied to the context. This big parameter is of course unrelated to the big parameter from the main tableau. Note that it is not a requirement for this parameter to be “big”, but it does allow for some code reuse in `isl`. In `PipLib`, the extra parameter is not “big”, but this may be because the big parameter of the main tableau also appears in the context tableau.

Finally, it was reported by Galea (2009), who worked on a parametric integer programming implementation in PPL (Bagnara et al.), that it is beneficial to add cuts for *all* rational coordinates in the context tableau. Based on this report, the initial `isl` implementation was adapted accordingly.

Generalized Basis Reduction

The default algorithm used in `isl` for feasibility checking is generalized basis reduction (Cook et al. 1991). This algorithm is also used in the `barvinok` implementation. The algorithm is fairly robust, but it has some overhead. We therefore try to avoid calling the algorithm in easy cases. In particular, we incrementally keep track of points for which the entire unit hypercube positioned at that point lies in the context. This set is described by translates of the constraints of the context and if (rationally) non-empty, any rational point in the set can be rounded up to yield an integer point in the context.

A restriction of the algorithm is that it only works on bounded sets. The affine hull of the recession cone therefore needs to be projected out first. As soon as the algorithm is invoked, we then also incrementally keep track of this recession cone. The reduced basis found by one call of the algorithm is also reused as initial basis for the next call.

Some problems lead to the introduction of many integer divisions. Within a given context, some of these integer divisions may be equal to each other, even if the expressions are not identical, or they may be equal to some affine combination of other variables. To detect such cases, we compute the affine hull of the context each time a new integer division is added. The algorithm used for computing this affine hull is that of Karr (1976), while the points used in this algorithm are obtained by performing integer feasibility checks on that part of the context outside the current approximation of the affine hull. The list of witnesses is used to construct an initial approximation of the hull, while any extra points found during the construction of the hull is added to this list. Any equality found in this way that expresses an integer division as an *integer* affine combination of other variables is propagated to the main tableau, where it is used to eliminate that integer division.

2.3.8 Experiments

Table 2.1 compares the execution times of `isl` (with both types of context tableau) on some more difficult instances to those of other tools, run on an Intel Xeon W3520 @ 2.66GHz. Easier problems such as the test cases distributed with `PipLib` can be solved so quickly that we would only be measuring overhead such as input/output and conversions and not the running time of the actual algorithm. We compare the following versions: `piplib-1.4.0-5-g0132fd9`, `barvinok-0.32.1-73-gc5d7751`, `isl-0.05.1-82-g3a37260` and PPL version 0.11.2.

	PipLib	barvinok	isl cut	isl gbr	PPL
Phideo	TC	793m	>999m	2.7s	372m
e1	0.33s	3.5s	0.08s	0.11s	0.18s
e3	0.14s	0.13s	0.10s	0.10s	0.17s
e4	0.24s	9.1s	0.09s	0.11s	0.70s
e5	0.12s	6.0s	0.06s	0.14s	0.17s
e6	0.10s	6.8s	0.17s	0.08s	0.21s
e7	0.03s	0.27s	0.04s	0.04s	0.03s
e8	0.03s	0.18s	0.03s	0.04s	0.01s
e9	OOM	70m	2.6s	0.94s	22s
vd	0.04s	0.10s	0.03s	0.03s	0.03s
bouleti	0.25s	line	0.06s	0.06s	0.15s
difficult	OOM	1.3s	1.7s	0.33s	1.4s
cnt/sum	TC	max	2.2s	2.2s	OOM
jcomplex	TC	max	3.7s	3.9s	OOM

Table 2.1: Comparison of Execution Times

The first test case is the following dependence analysis problem originating from the Phideo project (Verhaegh 1995) that was communicated to us by Bart Kienhuis:

```
lexmax { [j1,j2] -> [i1,i2,i3,i4,i5,i6,i7,i8,i9,i10] : 1 <= i1,j1
  <= 8 and 1 <= i2,i3,i4,i5,i6,i7,i8,i9,i10 <= 2 and 1 <= j2
  <= 128 and i1-1 = j1-1 and i2-1+2*i3-2+4*i4-4+8*i5-8+16*i6
  -16+32*i7-32+64*i8-64+128*i9-128+256*i10-256=3*j2-3+66 };
```

This problem was the main inspiration for some of the optimizations in Section 2.3.7. The second group of test cases are projections used during counting. The first nine of these come from Seghir and Loechner (2006). The remaining two come from Verdoolaege et al. (2005) and were used to drive the first, Gomory cuts based, implementation in *isl*. The third and final group of test cases are borrowed from Bygde (2010) and inspired the offline symmetry detection of Section 2.3.5. Without symmetry detection, the running times are 11s and 5.9s. All running times of *barvinok* and *isl* include a conversion to disjunctive normal form. Without this conversion, the final two cases can be solved in 0.07s and 0.21s. The *PipLib* implementation has some fixed limits and will sometimes report the problem to be too complex (TC), while on some other problems it will run out of memory (OOM). The *barvinok* implementation does not support problems with a non-trivial lineality space (line) nor maximization problems (max). The Gomory cuts based *isl* implementation was terminated after 1000 minutes on the first problem. The *gbr* version introduces some overhead on some of the easier problems, but is overall the clear winner.

2.3.9 Online Symmetry Detection

Manual experiments on small instances of the problems of Bygde (2010) and an analysis of the results by the approximate MPA method developed by Bygde (2010) have

revealed that these problems contain many more symmetries than can be detected using the offline method of Section 2.3.5. In this section, we present an online detection mechanism that has not been implemented yet, but that has shown promising results in manual applications.

Let us first consider what happens when we do not perform offline symmetry detection. At some point, one of the $b_i(\mathbf{p}) + \langle \mathbf{a}, \mathbf{x} \rangle \geq 0$ constraints, say the j th constraint, appears as a column variable, say c_1 , while the other constraints are represented as rows of the form $b_i(\mathbf{p}) - b_j(\mathbf{p}) + c$. The context is then split according to the relative order of $b_j(\mathbf{p})$ and one of the remaining $b_i(\mathbf{p})$. The offline method avoids this split by replacing all $b_i(\mathbf{p})$ by a single newly introduced parameter that represents the minimum of these $b_i(\mathbf{p})$. In the online method the split is similarly avoided by the introduction of a new parameter. In particular, a new parameter is introduced that represents $|b_j(\mathbf{p}) - b_i(\mathbf{p})|_+ = \max(b_j(\mathbf{p}) - b_i(\mathbf{p}), 0)$.

In general, let $r = b(\mathbf{p}) + \langle \mathbf{a}, \mathbf{c} \rangle$ be a row of the tableau such that the sign of $b(\mathbf{p})$ is indeterminate and such that exactly one of the elements of \mathbf{a} is a 1, while all remaining elements are non-positive. That is, $r = b(\mathbf{p}) + c_j - f$ with $f = -\sum_{i \neq j} a_i c_i \geq 0$. We introduce a new parameter t with context constraints $t \geq -b(\mathbf{p})$ and $t \geq 0$ and replace the column variable c_j by $c' + t$. The row r is now equal to $b(\mathbf{p}) + t + c' - f$. The constant term of this row is always non-negative because any negative value of $b(\mathbf{p})$ is compensated by $t \geq -b(\mathbf{p})$ while a non-negative value remains non-negative because $t \geq 0$.

We need to show that this transformation does not eliminate any valid solutions and that it does not introduce any spurious solutions. Given a valid solution for the original problem, we need to find a non-negative value of c' satisfying the constraints. If $b(\mathbf{p}) \geq 0$, we can take $t = 0$ so that $c' = c_j - t = c_j \geq 0$. If $b(\mathbf{p}) < 0$, we can take $t = -b(\mathbf{p})$. Since $r = b(\mathbf{p}) + c_j - f \geq 0$ and $f \geq 0$, we have $c' = c_j + b(\mathbf{p}) \geq 0$. Note that these choices amount to plugging in $t = |-b(\mathbf{p})|_+ = \max(-b(\mathbf{p}), 0)$. Conversely, given a solution to the new problem, we need to find a non-negative value of c_j , but this is easy since $c_j = c' + t$ and both of these are non-negative.

Plugging in $t = \max(-b(\mathbf{p}), 0)$ can be performed as in Section 2.3.6, but, as in the case of offline symmetry detection, it may be better to provide a direct representation for such expressions in the internal representation of sets and relations or at least in a quast-like output format.

2.4 Coalescing

See Verdoolaege (2009), for now. More details will be added later.

2.5 Transitive Closure

2.5.1 Introduction

Definition 2.5.1 (Power of a Relation) Let $R \in \mathbb{Z}^n \rightarrow 2^{\mathbb{Z}^{d+d}}$ be a relation and $k \in \mathbb{Z}_{\geq 1}$ a positive number, then power k of relation R is defined as

$$R^k := \begin{cases} R & \text{if } k = 1 \\ R \circ R^{k-1} & \text{if } k \geq 2. \end{cases} \quad (2.1)$$

Definition 2.5.2 (Transitive Closure of a Relation) Let $R \in \mathbb{Z}^n \rightarrow 2^{\mathbb{Z}^{d+d}}$ be a relation, then the transitive closure R^+ of R is the union of all positive powers of R ,

$$R^+ := \bigcup_{k \geq 1} R^k.$$

Alternatively, the transitive closure may be defined inductively as

$$R^+ := R \cup (R \circ R^+). \quad (2.2)$$

Since the transitive closure of a polyhedral relation may no longer be a polyhedral relation (Kelly et al. 1996c), we can, in the general case, only compute an approximation of the transitive closure. Whereas Kelly et al. (1996c) compute underapproximations, we, like Beletska et al. (2009), compute overapproximations. That is, given a relation R , we will compute a relation T such that $R^+ \subseteq T$. Of course, we want this approximation to be as close as possible to the actual transitive closure R^+ and we want to detect the cases where the approximation is exact, i.e., where $T = R^+$.

For computing an approximation of the transitive closure of R , we follow the same general strategy as Beletska et al. (2009) and first compute an approximation of R^k for $k \geq 1$ and then project out the parameter k from the resulting relation.

Example 2.5.3 As a trivial example, consider the relation $R = \{x \rightarrow x + 1\}$. The k th power of this map for arbitrary k is

$$R^k = k \mapsto \{x \rightarrow x + k \mid k \geq 1\}.$$

The transitive closure is then

$$\begin{aligned} R^+ &= \{x \rightarrow y \mid \exists k \in \mathbb{Z}_{\geq 1} : y = x + k\} \\ &= \{x \rightarrow y \mid y \geq x + 1\}. \end{aligned}$$

2.5.2 Computing an Approximation of R^k

There are some special cases where the computation of R^k is very easy. One such case is that where R does not compose with itself, i.e., $R \circ R = \emptyset$ or $\text{dom } R \cap \text{ran } R = \emptyset$. In this case, R^k is only non-empty for $k = 1$ where it is equal to R itself.

In general, it is impossible to construct a closed form of R^k as a polyhedral relation. We will therefore need to make some approximations. As a first approximations, we

will consider each of the basic relations in R as simply adding one or more offsets to a domain element to arrive at an image element and ignore the fact that some of these offsets may only be applied to some of the domain elements. That is, we will only consider the difference set ΔR of the relation. In particular, we will first construct a collection P of paths that move through a total of k offsets and then intersect domain and range of this collection with those of R . That is,

$$K = P \cap (\text{dom } R \rightarrow \text{ran } R), \quad (2.3)$$

with

$$P = \mathbf{s} \mapsto \{ \mathbf{x} \rightarrow \mathbf{y} \mid \exists k_i \in \mathbb{Z}_{\geq 0}, \delta_i \in k_i \Delta_i(\mathbf{s}) : \mathbf{y} = \mathbf{x} + \sum_i \delta_i \wedge \sum_i k_i = k > 0 \} \quad (2.4)$$

and with Δ_i the basic sets that compose the difference set ΔR . Note that the number of basic sets Δ_i need not be the same as the number of basic relations in R . Also note that since addition is commutative, it does not matter in which order we add the offsets and so we are allowed to group them as we did in (2.4).

If all the Δ_i s are singleton sets $\Delta_i = \{ \delta_i \}$ with $\delta_i \in \mathbb{Z}^d$, then (2.4) simplifies to

$$P = \{ \mathbf{x} \rightarrow \mathbf{y} \mid \exists k_i \in \mathbb{Z}_{\geq 0} : \mathbf{y} = \mathbf{x} + \sum_i k_i \delta_i \wedge \sum_i k_i = k > 0 \} \quad (2.5)$$

and then the approximation computed in (2.3) is essentially the same as that of Beletska et al. (2009). If some of the Δ_i s are not singleton sets or if some of δ_i s are parametric, then we need to resort to further approximations.

To ease both the exposition and the implementation, we will for the remainder of this section work with extended offsets $\Delta'_i = \Delta_i \times \{1\}$. That is, each offset is extended with an extra coordinate that is set equal to one. The paths constructed by summing such extended offsets have the length encoded as the difference of their final coordinates. The path P' can then be decomposed into paths P'_i , one for each Δ_i ,

$$P' = ((P'_m \cup \text{Id}) \circ \dots \circ (P'_2 \cup \text{Id}) \circ (P'_1 \cup \text{Id})) \cap \{ \mathbf{x}' \rightarrow \mathbf{y}' \mid y_{d+1} - x_{d+1} = k > 0 \}, \quad (2.6)$$

with

$$P'_i = \mathbf{s} \mapsto \{ \mathbf{x}' \rightarrow \mathbf{y}' \mid \exists k \in \mathbb{Z}_{\geq 1}, \delta \in k \Delta'_i(\mathbf{s}) : \mathbf{y}' = \mathbf{x}' + \delta \}.$$

Note that each P'_i contains paths of length at least one. We therefore need to take the union with the identity relation when composing the P'_i s to allow for paths that do not contain any offsets from one or more Δ'_i . The path that consists of only identity relations is removed by imposing the constraint $y_{d+1} - x_{d+1} > 0$. Taking the union with the identity relation means that the relations we compose in (2.6) each consist of two basic relations. If there are m disjuncts in the input relation, then a direct application of the composition operation may therefore result in a relation with 2^m disjuncts, which is prohibitively expensive. It is therefore crucial to apply coalescing (Section 2.4) after each composition.

Let us now consider how to compute an overapproximation of P'_i . Those that correspond to singleton Δ_i s are grouped together and handled as in (2.5). Note that this is

just an optimization. The procedure described below would produce results that are at least as accurate. For simplicity, we first assume that no constraint in Δ'_i involves any existentially quantified variables. We will return to existentially quantified variables at the end of this section. Without existentially quantified variables, we can classify the constraints of Δ'_i as follows

1. non-parametric constraints

$$A_1 \mathbf{x} + \mathbf{c}_1 \geq \mathbf{0} \quad (2.7)$$

2. purely parametric constraints

$$B_2 \mathbf{s} + \mathbf{c}_2 \geq \mathbf{0} \quad (2.8)$$

3. negative mixed constraints

$$A_3 \mathbf{x} + B_3 \mathbf{s} + \mathbf{c}_3 \geq \mathbf{0} \quad (2.9)$$

such that for each row j and for all \mathbf{s} ,

$$\Delta'_i(\mathbf{s}) \cap \{ \delta' \mid B_{3,j} \mathbf{s} + c_{3,j} > 0 \} = \emptyset$$

4. positive mixed constraints

$$A_4 \mathbf{x} + B_4 \mathbf{s} + \mathbf{c}_4 \geq \mathbf{0}$$

such that for each row j , there is at least one \mathbf{s} such that

$$\Delta'_i(\mathbf{s}) \cap \{ \delta' \mid B_{4,j} \mathbf{s} + c_{4,j} > 0 \} \neq \emptyset$$

We will use the following approximation Q_i for P'_i :

$$Q_i = \mathbf{s} \mapsto \{ \mathbf{x}' \rightarrow \mathbf{y}' \mid \exists k \in \mathbb{Z}_{\geq 1}, \mathbf{f} \in \mathbb{Z}^d : \mathbf{y}' = \mathbf{x}' + (\mathbf{f}, k) \wedge A_1 \mathbf{f} + k \mathbf{c}_1 \geq \mathbf{0} \wedge B_2 \mathbf{s} + \mathbf{c}_2 \geq \mathbf{0} \wedge A_3 \mathbf{f} + B_3 \mathbf{s} + \mathbf{c}_3 \geq \mathbf{0} \}. \quad (2.10)$$

To prove that Q_i is indeed an overapproximation of P'_i , we need to show that for every $\mathbf{s} \in \mathbb{Z}^n$, for every $k \in \mathbb{Z}_{\geq 1}$ and for every $\mathbf{f} \in k \Delta_i(\mathbf{s})$ we have that (\mathbf{f}, k) satisfies the constraints in (2.10). If $\Delta_i(\mathbf{s})$ is non-empty, then \mathbf{s} must satisfy the constraints in (2.8). Each element $(\mathbf{f}, k) \in k \Delta'_i(\mathbf{s})$ is a sum of k elements $(\mathbf{f}_j, 1)$ in $\Delta'_i(\mathbf{s})$. Each of these elements satisfies the constraints in (2.7), i.e.,

$$\begin{bmatrix} A_1 & \mathbf{c}_1 \end{bmatrix} \begin{bmatrix} \mathbf{f}_j \\ 1 \end{bmatrix} \geq \mathbf{0}.$$

The sum of these elements therefore satisfies the same set of inequalities, i.e., $A_1 \mathbf{f} + k \mathbf{c}_1 \geq \mathbf{0}$. Finally, the constraints in (2.9) are such that for any \mathbf{s} in the parameter domain of Δ , we have $-\mathbf{r}(\mathbf{s}) := B_3 \mathbf{s} + \mathbf{c}_3 \leq \mathbf{0}$, i.e., $A_3 \mathbf{f}_j \geq \mathbf{r}(\mathbf{s}) \geq \mathbf{0}$ and therefore also $A_3 \mathbf{f} \geq \mathbf{r}(\mathbf{s})$. Note that if there are no mixed constraints and if the rational relaxation of $\Delta_i(\mathbf{s})$, i.e., $\{ \mathbf{x} \in \mathbb{Q}^d \mid A_1 \mathbf{x} + \mathbf{c}_1 \geq \mathbf{0} \}$, has integer vertices, then the approximation

is exact, i.e., $Q_i = P'_i$. In this case, the vertices of $\Delta'_i(\mathbf{s})$ generate the rational cone $\{\mathbf{x}' \in \mathbb{Q}^{d+1} \mid [A_1 \quad \mathbf{c}_1] \mathbf{x}'\}$ and therefore $\Delta'_i(\mathbf{s})$ is a Hilbert basis of this cone (Schrijver 1986, Theorem 16.4).

Note however that, as pointed out by De Smet (2010), if there *are* any mixed constraints, then the above procedure may not compute the most accurate affine approximation of $k \Delta_i(\mathbf{s})$ with $k \geq 1$. In particular, we only consider the negative mixed constraints that happen to appear in the description of $\Delta_i(\mathbf{s})$, while we should instead consider *all* valid such constraints. It is also sufficient to consider those constraints because any constraint that is valid for $k \Delta_i(\mathbf{s})$ is also valid for $1 \Delta_i(\mathbf{s}) = \Delta_i(\mathbf{s})$. Take therefore any constraint $\langle \mathbf{a}, \mathbf{x} \rangle + \langle \mathbf{b}, \mathbf{s} \rangle + c \geq 0$ valid for $\Delta_i(\mathbf{s})$. This constraint is also valid for $k \Delta_i(\mathbf{s})$ iff $k \langle \mathbf{a}, \mathbf{x} \rangle + \langle \mathbf{b}, \mathbf{s} \rangle + c \geq 0$. If $\langle \mathbf{b}, \mathbf{s} \rangle + c$ can attain any positive value, then $\langle \mathbf{a}, \mathbf{x} \rangle$ may be negative for some elements of $\Delta_i(\mathbf{s})$. We then have $k \langle \mathbf{a}, \mathbf{x} \rangle < \langle \mathbf{a}, \mathbf{x} \rangle$ for $k > 1$ and so the constraint is not valid for $k \Delta_i(\mathbf{s})$. We therefore need to impose $\langle \mathbf{b}, \mathbf{s} \rangle + c \leq 0$ for all values of \mathbf{s} such that $\Delta_i(\mathbf{s})$ is non-empty, i.e., \mathbf{b} and c need to be such that $-\langle \mathbf{b}, \mathbf{s} \rangle - c \geq 0$ is a valid constraint of $\Delta_i(\mathbf{s})$. That is, (\mathbf{b}, c) are the opposites of the coefficients of a valid constraint of $\Delta_i(\mathbf{s})$. The approximation of $k \Delta_i(\mathbf{s})$ can therefore be obtained using three applications of Farkas' lemma. The first obtains the coefficients of constraints valid for $\Delta_i(\mathbf{s})$. The second obtains the coefficients of constraints valid for the projection of $\Delta_i(\mathbf{s})$ onto the parameters. The opposite of the second set is then computed and intersected with the first set. The result is the set of coefficients of constraints valid for $k \Delta_i(\mathbf{s})$. A final application of Farkas' lemma is needed to obtain the approximation of $k \Delta_i(\mathbf{s})$ itself.

Example 2.5.4 Consider the relation

$$n \rightarrow \{(x, y) \rightarrow (1 + x, 1 - n + y) \mid n \geq 2\}.$$

The difference set of this relation is

$$\Delta = n \rightarrow \{(1, 1 - n) \mid n \geq 2\}.$$

Using our approach, we would only consider the mixed constraint $y - 1 + n \geq 0$, leading to the following approximation of the transitive closure:

$$n \rightarrow \{(x, y) \rightarrow (o_0, o_1) \mid n \geq 2 \wedge o_1 \leq 1 - n + y \wedge o_0 \geq 1 + x\}.$$

If, instead, we apply Farkas's lemma to Δ , i.e.,

```
D := [n] -> { [1, 1 - n] : n >= 2 };
CD := coefficients D;
CD;
```

we obtain

```
{ rat: coefficients[[c_cst, c_n] -> [i2, i3]] : i3 <= c_n and
  i3 <= c_cst + 2c_n + i2 }
```

The pure-parametric constraints valid for Δ ,

```

P := { [a,b] -> [] }(D);
CP := coefficients P;
CP;

```

are

```

{ rat: coefficients[[c_cst, c_n] -> []] : c_n >= 0 and 2c_n >= -c_cst }

```

Negating these coefficients and intersecting with CD,

```

NCP := { rat: coefficients[[a,b] -> []]
        -> coefficients[[-a,-b] -> []] }(CP);
CK := wrap((unwrap CD) * (dom (unwrap NCP)));
CK;

```

we obtain

```

{ rat: [[c_cst, c_n] -> [i2, i3]] : i3 <= c_n and
  i3 <= c_cst + 2c_n + i2 and c_n <= 0 and 2c_n <= -c_cst }

```

The approximation for $k\Delta$,

```

K := solutions CK;
K;

```

is then

```

[n] -> { rat: [i0, i1] : i1 <= -i0 and i0 >= 1 and i1 <= 2 - n - i0 }

```

Finally, the computed approximation for R^+ ,

```

T := unwrap({ [dx,dy] -> [[x,y] -> [x+dx,y+dy]] }(K));
R := [n] -> { [x,y] -> [x+1,y+1-n] : n >= 2 };
T := T * ((dom R) -> (ran R));
T;

```

is

```

[n] -> { [x, y] -> [o0, o1] : o1 <= x + y - o0 and
  o0 >= 1 + x and o1 <= 2 - n + x + y - o0 and n >= 2 }

```

Existentially quantified variables can be handled by classifying them into variables that are uniquely determined by the parameters, variables that are independent of the parameters and others. The first set can be treated as parameters and the second as variables. Constraints involving the other existentially quantified variables are removed.

Example 2.5.5 *Consider the relation*

$$R = n \rightarrow \{ x \rightarrow y \mid \exists \alpha_0, \alpha_1 : 7\alpha_0 = -2 + n \wedge 5\alpha_1 = -1 - x + y \wedge y \geq 6 + x \}.$$

The difference set of this relation is

$$\Delta = \Delta R = n \rightarrow \{ x \mid \exists \alpha_0, \alpha_1 : 7\alpha_0 = -2 + n \wedge 5\alpha_1 = -1 + x \wedge x \geq 6 \}.$$

The existentially quantified variables can be defined in terms of the parameters and variables as

$$\alpha_0 = \left\lfloor \frac{-2+n}{7} \right\rfloor \quad \text{and} \quad \alpha_1 = \left\lfloor \frac{-1+x}{5} \right\rfloor.$$

α_0 can therefore be treated as a parameter, while α_1 can be treated as a variable. This in turn means that $7\alpha_0 = -2+n$ can be treated as a purely parametric constraint, while the other two constraints are non-parametric. The corresponding Q (2.10) is therefore

$$n \rightarrow \{ (x, z) \rightarrow (y, w) \mid \exists \alpha_0, \alpha_1, k, f : k \geq 1 \wedge y = x + f \wedge w = z + k \wedge 7\alpha_0 = -2 + n \wedge 5\alpha_1 = -k + x \wedge x \geq 6k \}.$$

Projecting out the final coordinates encoding the length of the paths, results in the exact transitive closure

$$R^+ = n \rightarrow \{ x \rightarrow y \mid \exists \alpha_0, \alpha_1 : 7\alpha_1 = -2 + n \wedge 6\alpha_0 \geq -x + y \wedge 5\alpha_0 \leq -1 - x + y \}.$$

The fact that we ignore some impure constraints clearly leads to a loss of accuracy. In some cases, some of this loss can be recovered by not considering the parameters in a special way. That is, instead of considering the set

$$\Delta = \Delta R = \mathbf{s} \mapsto \{ \delta \in \mathbb{Z}^d \mid \exists \mathbf{x} \rightarrow \mathbf{y} \in R : \delta = \mathbf{y} - \mathbf{x} \}$$

we consider the set

$$\Delta' = \Delta R' = \{ \delta \in \mathbb{Z}^{n+d} \mid \exists (\mathbf{s}, \mathbf{x}) \rightarrow (\mathbf{s}, \mathbf{y}) \in R' : \delta = (\mathbf{s} - \mathbf{s}, \mathbf{y} - \mathbf{x}) \}.$$

The first n coordinates of every element in Δ' are zero. Projecting out these zero coordinates from Δ' is equivalent to projecting out the parameters in Δ . The result is obviously a superset of Δ , but all its constraints are of type (2.7) and they can therefore all be used in the construction of Q_i .

Example 2.5.6 Consider the relation

$$R = n \rightarrow \{ (x, y) \rightarrow (1 + x, 1 - n + y) \mid n \geq 2 \}.$$

We have

$$\Delta R = n \rightarrow \{ (1, 1 - n) \mid n \geq 2 \}$$

and so, by treating the parameters in a special way, we obtain the following approximation for R^+ :

$$n \rightarrow \{ (x, y) \rightarrow (x', y') \mid n \geq 2 \wedge y' \leq 1 - n + y \wedge x' \geq 1 + x \}.$$

If we consider instead

$$R' = \{ (n, x, y) \rightarrow (n, 1 + x, 1 - n + y) \mid n \geq 2 \}$$

then

$$\Delta R' = \{ (0, 1, y) \mid y \leq -1 \}$$

and we obtain the approximation

$$n \rightarrow \{ (x, y) \rightarrow (x', y') \mid n \geq 2 \wedge x' \geq 1 + x \wedge y' \leq x + y - x' \}.$$

If we consider both ΔR and $\Delta R'$, then we obtain

$$n \rightarrow \{ (x, y) \rightarrow (x', y') \mid n \geq 2 \wedge y' \leq 1 - n + y \wedge x' \geq 1 + x \wedge y' \leq x + y - x' \}.$$

Note, however, that this is not the most accurate affine approximation that can be obtained. That would be

$$n \rightarrow \{ (x, y) \rightarrow (x', y') \mid y' \leq 2 - n + x + y - x' \wedge n \geq 2 \wedge x' \geq 1 + x \}.$$

2.5.3 Checking Exactness

The approximation T for the transitive closure R^+ can be obtained by projecting out the parameter k from the approximation K (2.3) of the power R^k . Since K is an overapproximation of R^k , T will also be an overapproximation of R^+ . To check whether the results are exact, we need to consider two cases depending on whether R is *cyclic*, where R is defined to be cyclic if R^+ maps any element to itself, i.e., $R^+ \cap \text{Id} \neq \emptyset$. If R is acyclic, then the inductive definition of (2.2) is equivalent to its completion, i.e.,

$$R^+ = R \cup (R \circ R^+)$$

is a defining property. Since T is known to be an overapproximation, we only need to check whether

$$T \subseteq R \cup (R \circ T).$$

This is essentially Theorem 5 of Kelly et al. (1996c). The only difference is that they only consider lexicographically forward relations, a special case of acyclic relations.

If, on the other hand, R is cyclic, then we have to resort to checking whether the approximation K of the power is exact. Note that T may be exact even if K is not exact, so the check is sound, but incomplete. To check exactness of the power, we simply need to check (2.1). Since again K is known to be an overapproximation, we only need to check whether

$$\begin{aligned} K'|_{y_{d+1}-x_{d+1}=1} &\subseteq R' \\ K'|_{y_{d+1}-x_{d+1} \geq 2} &\subseteq R' \circ K'|_{y_{d+1}-x_{d+1} \geq 1}, \end{aligned}$$

where $R' = \{ \mathbf{x}' \rightarrow \mathbf{y}' \mid \mathbf{x} \rightarrow \mathbf{y} \in R \wedge y_{d+1} - x_{d+1} = 1 \}$, i.e., R extended with path lengths equal to 1.

All that remains is to explain how to check the cyclicity of R . Note that the exactness on the power is always sound, even in the acyclic case, so we only need to be careful that we find all cyclic cases. Now, if R is cyclic, i.e., $R^+ \cap \text{Id} \neq \emptyset$, then, since T is an overapproximation of R^+ , also $T \cap \text{Id} \neq \emptyset$. This in turn means that $\Delta K'$ contains a point whose first d coordinates are zero and whose final coordinate is positive. In the implementation we currently perform this test on P' instead of K' . Note that if R^+ is acyclic and T is not, then the approximation is clearly not exact and the approximation of the power K will not be exact either.

2.5.4 Decomposing R into strongly connected components

If the input relation R is a union of several basic relations that can be partially ordered then the accuracy of the approximation may be improved by computing an approximation of each strongly connected components separately. For example, if $R = R_1 \cup R_2$ and $R_1 \circ R_2 = \emptyset$, then we know that any path that passes through R_2 cannot later pass through R_1 , i.e.,

$$R^+ = R_1^+ \cup R_2^+ \cup (R_2^+ \circ R_1^+). \quad (2.11)$$

We can therefore compute (approximations of) transitive closures of R_1 and R_2 separately. Note, however, that the condition $R_1 \circ R_2 = \emptyset$ is actually too strong. If $R_1 \circ R_2$ is a subset of $R_2 \circ R_1$ then we can reorder the segments in any path that moves through both R_1 and R_2 to first move through R_1 and then through R_2 .

This idea can be generalized to relations that are unions of more than two basic relations by constructing the strongly connected components in the graph with as vertices the basic relations and an edge between two basic relations R_i and R_j if R_i needs to follow R_j in some paths. That is, there is an edge from R_i to R_j iff

$$R_i \circ R_j \not\subseteq R_j \circ R_i. \quad (2.12)$$

The components can be obtained from the graph by applying Tarjan's algorithm (Tarjan 1972).

In practice, we compute the (extended) powers K'_i of each component separately and then compose them as in (2.6). Note, however, that in this case the order in which we apply them is important and should correspond to a topological ordering of the strongly connected components. Simply applying Tarjan's algorithm will produce topologically sorted strongly connected components. The graph on which Tarjan's algorithm is applied is constructed on-the-fly. That is, whenever the algorithm checks if there is an edge between two vertices, we evaluate (2.12). The exactness check is performed on each component separately. If the approximation turns out to be inexact for any of the components, then the entire result is marked inexact and the exactness check is skipped on the components that still need to be handled.

It should be noted that (2.11) is only valid for exact transitive closures. If overapproximations are computed in the right hand side, then the result will still be an overapproximation of the left hand side, but this result may not be transitively closed. If we only separate components based on the condition $R_i \circ R_j = \emptyset$, then there is no problem, as this condition will still hold on the computed approximations of the transitive closures. If, however, we have exploited (2.12) during the decomposition and if the result turns out not to be exact, then we check whether the result is transitively closed. If not, we recompute the transitive closure, skipping the decomposition. Note that testing for transitive closedness on the result may be fairly expensive, so we may want to make this check configurable.

Example 2.5.7 Consider the relation in example `closure4` that comes with the Omega calculator (Kelly et al. 1996a), $R = R_1 \cup R_2$, with

$$\begin{aligned} R_1 &= \{ (x, y) \rightarrow (x, y + 1) \mid 1 \leq x, y \leq 10 \} \\ R_2 &= \{ (x, y) \rightarrow (x + 1, y) \mid 1 \leq x \leq 20 \wedge 5 \leq y \leq 15 \}. \end{aligned}$$

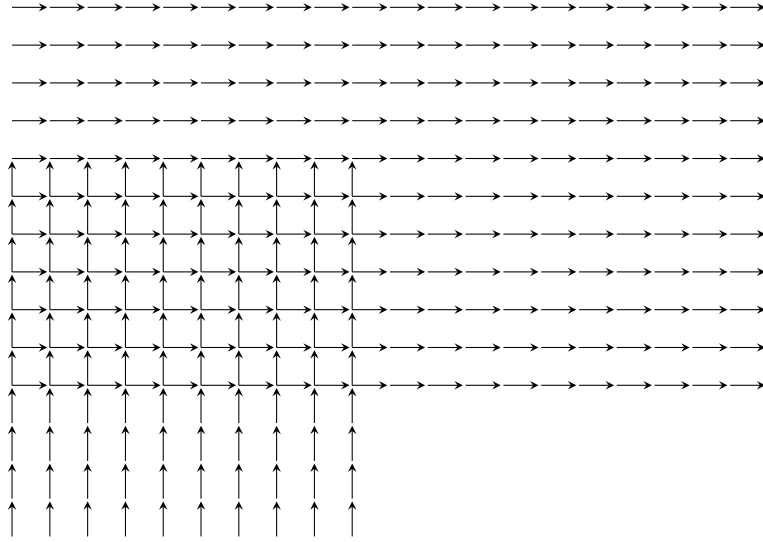


Figure 2.1: The relation from Example 2.5.7

This relation is shown graphically in Figure 2.1. We have

$$R_1 \circ R_2 = \{ (x, y) \rightarrow (x + 1, y + 1) \mid 1 \leq x \leq 9 \wedge 5 \leq y \leq 10 \}$$

$$R_2 \circ R_1 = \{ (x, y) \rightarrow (x + 1, y + 1) \mid 1 \leq x \leq 10 \wedge 4 \leq y \leq 10 \}.$$

Clearly, $R_1 \circ R_2 \subseteq R_2 \circ R_1$ and so

$$(R_1 \cup R_2)^+ = (R_2^+ \circ R_1^+) \cup R_1^+ \cup R_2^+.$$

Example 2.5.8 Consider the relation on the right of Beletska et al. (2009, Figure 2), reproduced in Figure 2.2. The relation can be described as $R = R_1 \cup R_2 \cup R_3$, with

$$R_1 = n \mapsto \{ (i, j) \rightarrow (i + 3, j) \mid i \leq 2j - 4 \wedge i \leq n - 3 \wedge j \leq 2i - 1 \wedge j \leq n \}$$

$$R_2 = n \mapsto \{ (i, j) \rightarrow (i, j + 3) \mid i \leq 2j - 1 \wedge i \leq n \wedge j \leq 2i - 4 \wedge j \leq n - 3 \}$$

$$R_3 = n \mapsto \{ (i, j) \rightarrow (i + 1, j + 1) \mid i \leq 2j - 1 \wedge i \leq n - 1 \wedge j \leq 2i - 1 \wedge j \leq n - 1 \}.$$

The figure shows this relation for $n = 7$. Both $R_3 \circ R_1 \subseteq R_1 \circ R_3$ and $R_3 \circ R_2 \subseteq R_2 \circ R_3$, which the reader can verify using the `iscc` calculator:

```

R1 := [n] -> { [i, j] -> [i+3, j] : i <= 2 j - 4 and i <= n - 3 and
                                     j <= 2 i - 1 and j <= n };
R2 := [n] -> { [i, j] -> [i, j+3] : i <= 2 j - 1 and i <= n and
                                     j <= 2 i - 4 and j <= n - 3 };
R3 := [n] -> { [i, j] -> [i+1, j+1] : i <= 2 j - 1 and i <= n - 1 and
                                     j <= 2 i - 1 and j <= n - 1 };

(R1 . R3) - (R3 . R1);
(R2 . R3) - (R3 . R2);

```

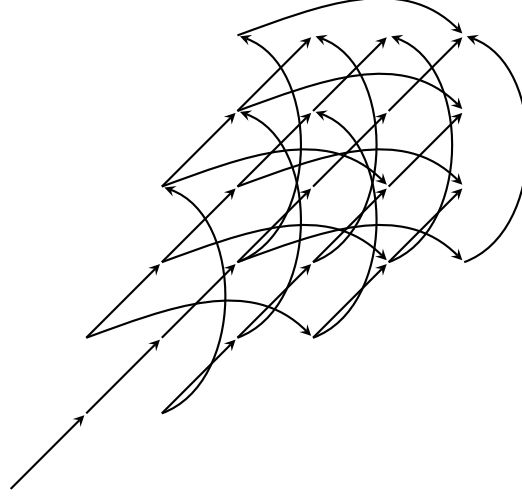


Figure 2.2: The relation from Example 2.5.8

R_3 can therefore be moved forward in any path. For the other two basic relations, we have both $R_2 \circ R_1 \not\subseteq R_1 \circ R_2$ and $R_1 \circ R_2 \not\subseteq R_2 \circ R_1$ and so R_1 and R_2 form a strongly connected component. By computing the power of R_3 and $R_1 \cup R_2$ separately and composing the results, the power of R can be computed exactly using (2.5). As explained by Beletska et al. (2009), applying the same formula to R directly, without a decomposition, would result in an overapproximation of the power.

2.5.5 Partitioning the domains and ranges of R

The algorithm of Section 2.5.2 assumes that the input relation R can be treated as a union of translations. This is a reasonable assumption if R maps elements of a given abstract domain to the same domain. However, if R is a union of relations that map between different domains, then this assumption no longer holds. In particular, when an entire dependence graph is encoded in a single relation, as is done by, e.g., Barthou et al. (2000, Section 6.1), then it does not make sense to look at differences between iterations of different domains. Now, arguably, a modified Floyd-Warshall algorithm should be applied to the dependence graph, as advocated by Kelly et al. (1996c), with the transitive closure operation only being applied to relations from a given domain to itself. However, it is also possible to detect disjoint domains and ranges and to apply Floyd-Warshall internally.

Let the input relation R be a union of m basic relations R_i . Let D_{2i} be the domains of R_i and D_{2i+1} the ranges of R_i . The first step is to group overlapping D_j until a partition is obtained. If the resulting partition consists of a single part, then we continue with the algorithm of Section 2.5.2. Otherwise, we apply Floyd-Warshall on the graph with as vertices the parts of the partition and as edges the R_i attached to the appropriate pairs of

Algorithm 1: The modified Floyd-Warshall algorithm of Kelly et al. (1996c)

Input: Relations R_{pq} , $0 \leq p, q < n$

Output: Updated relations R_{pq} such that each relation R_{pq} contains all indirect paths from p to q in the input graph

```

1 for  $r \in [0, n - 1]$  do
2    $R_{rr} := R_{rr}^+$ 
3   for  $p \in [0, n - 1]$  do
4     for  $q \in [0, n - 1]$  do
5       if  $p \neq r$  or  $q \neq r$  then
6          $R_{pq} := R_{pq} \cup (R_{rq} \circ R_{pr}) \cup (R_{rq} \circ R_{rr} \circ R_{pr})$ 

```

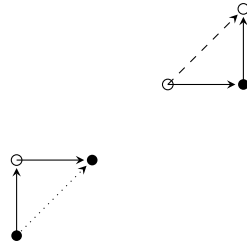


Figure 2.3: The relation (solid arrows) on the right of Figure 1 of Beletskaya et al. (2009) and its transitive closure

vertices. In particular, let there be n parts P_k in the partition. We construct n^2 relations

$$R_{pq} := \bigcup_{i \text{ s.t. } \text{dom } R_i \subseteq P_p \wedge \text{ran } R_i \subseteq P_q} R_i,$$

apply algorithm 1 and return the union of all resulting R_{pq} as the transitive closure of R . Each iteration of the r -loop in algorithm 1 updates all relations R_{pq} to include paths that go from p to r , possibly stay there for a while, and then go from r to q . Note that paths that “stay in r ” include all paths that pass through earlier vertices since R_{rr} itself has been updated accordingly in previous iterations of the outer loop. In principle, it would be sufficient to use the R_{pr} and R_{rq} computed in the previous iteration of the r -loop in Line 6. However, from an implementation perspective, it is easier to allow either or both of these to have been updated in the same iteration of the r -loop. This may result in duplicate paths, but these can usually be removed by coalescing (Section 2.4) the result of the union in Line 6, which should be done in any case. The transitive closure in Line 2 is performed using a recursive call. This recursive call includes the partitioning step, but the resulting partition will usually be a singleton. The result of the recursive call will either be exact or an overapproximation. The final result of Floyd-Warshall is therefore also exact or an overapproximation.

Example 2.5.9 Consider the relation on the right of Figure 1 of Beletskaya et al. (2009),

reproduced in Figure 2.3. This relation can be described as

$$\{(x, y) \rightarrow (x_2, y_2) \mid (3y = 2x \wedge x_2 = x \wedge 3y_2 = 3 + 2x \wedge x \geq 0 \wedge x \leq 3) \vee (x_2 = 1 + x \wedge y_2 = y \wedge x \geq 0 \wedge 3y \geq 2 + 2x \wedge x \leq 2 \wedge 3y \leq 3 + 2x)\}.$$

Note that the domain of the upward relation overlaps with the range of the rightward relation and vice versa, but that the domain of neither relation overlaps with its own range or the domain of the other relation. The domains and ranges can therefore be partitioned into two parts, P_0 and P_1 , shown as the white and black dots in Figure 2.3, respectively. Initially, we have

$$\begin{aligned} R_{00} &= \emptyset \\ R_{01} &= \{(x, y) \rightarrow (x + 1, y) \mid (x \geq 0 \wedge 3y \geq 2 + 2x \wedge x \leq 2 \wedge 3y \leq 3 + 2x)\} \\ R_{10} &= \{(x, y) \rightarrow (x_2, y_2) \mid (3y = 2x \wedge x_2 = x \wedge 3y_2 = 3 + 2x \wedge x \geq 0 \wedge x \leq 3)\} \\ R_{11} &= \emptyset. \end{aligned}$$

In the first iteration, R_{00} remains the same ($\emptyset^+ = \emptyset$). R_{01} and R_{10} are therefore also unaffected, but R_{11} is updated to include $R_{01} \circ R_{10}$, i.e., the dashed arrow in the figure. This new R_{11} is obviously transitively closed, so it is not changed in the second iteration and it does not have an effect on R_{01} and R_{10} . However, R_{00} is updated to include $R_{10} \circ R_{01}$, i.e., the dotted arrow in the figure. The transitive closure of the original relation is then equal to $R_{00} \cup R_{01} \cup R_{10} \cup R_{11}$.

2.5.6 Incremental Computation

In some cases it is possible and useful to compute the transitive closure of union of basic relations incrementally. In particular, if R is a union of m basic maps,

$$R = \bigcup_j R_j,$$

then we can pick some R_i and compute the transitive closure of R as

$$R^+ = R_i^+ \cup \left(\bigcup_{j \neq i} R_i^* \circ R_j \circ R_i^* \right)^+. \quad (2.13)$$

For this approach to be successful, it is crucial that each of the disjuncts in the argument of the second transitive closure in (2.13) be representable as a single basic relation, i.e., without a union. If this condition holds, then by using (2.13), the number of disjuncts in the argument of the transitive closure can be reduced by one. Now, $R_i^* = R_i^+ \cup \text{Id}$, but in some cases it is possible to relax the constraints of R_i^+ to include part of the identity relation, say on domain D . We will use the notation $C(R_i, D) = R_i^+ \cup \text{Id}_D$ to represent this relaxed version of R^+ . Kelly et al. (1996c) use the notation $R_i^?$. $C(R_i, D)$ can be computed by allowing k to attain the value 0 in (2.10) and by using

$$P \cap (D \rightarrow D)$$

instead of (2.3). Typically, D will be a strict superset of both $\text{dom } R_i$ and $\text{ran } R_i$. We therefore need to check that domain and range of the transitive closure are part of $C(R_i, D)$, i.e., the part that results from the paths of positive length ($k \geq 1$), are equal to the domain and range of R_i . If not, then the incremental approach cannot be applied for the given choice of R_i and D .

In order to be able to replace R^* by $C(R_i, D)$ in (2.13), D should be chosen to include both $\text{dom } R$ and $\text{ran } R$, i.e., such that $\text{Id}_D \circ R_j \circ \text{Id}_D = R_j$ for all $j \neq i$. Kelly et al. (1996c) say that they use $D = \text{dom } R_i \cup \text{ran } R_i$, but presumably they mean that they use $D = \text{dom } R \cup \text{ran } R$. Now, this expression of D contains a union, so it not directly usable. Kelly et al. (1996c) do not explain how they avoid this union. Apparently, in their implementation, they are using the convex hull of $\text{dom } R \cup \text{ran } R$ or at least an approximation of this convex hull. We use the simple hull (Section 2.2) of $\text{dom } R \cup \text{ran } R$.

It is also possible to use a domain D that does *not* include $\text{dom } R \cup \text{ran } R$, but then we have to compose with $C(R_i, D)$ more selectively. In particular, if we have

$$\text{for each } j \neq i \text{ either } \text{dom } R_j \subseteq D \text{ or } \text{dom } R_j \cap \text{ran } R_i = \emptyset \quad (2.14)$$

and, similarly,

$$\text{for each } j \neq i \text{ either } \text{ran } R_j \subseteq D \text{ or } \text{ran } R_j \cap \text{dom } R_i = \emptyset \quad (2.15)$$

then we can refine (2.13) to

$$R_i^+ \cup \left(\left(\bigcup_{\substack{\text{dom } R_j \subseteq D \\ \text{ran } R_j \subseteq D}} C \circ R_j \circ C \right) \cup \left(\bigcup_{\substack{\text{dom } R_j \cap \text{ran } R_i = \emptyset \\ \text{ran } R_j \subseteq D}} C \circ R_j \right) \cup \left(\bigcup_{\substack{\text{dom } R_j \subseteq D \\ \text{ran } R_j \cap \text{dom } R_i = \emptyset}} R_j \circ C \right) \cup \left(\bigcup_{\substack{\text{dom } R_j \cap \text{ran } R_i = \emptyset \\ \text{ran } R_j \cap \text{dom } R_i = \emptyset}} R_j \right) \right)^+.$$

If only property (2.14) holds, we can use

$$R_i^+ \cup \left((R_i^+ \cup \text{Id}) \circ \left(\left(\bigcup_{\text{dom } R_j \subseteq D} R_j \circ C \right) \cup \left(\bigcup_{\text{dom } R_j \cap \text{ran } R_i = \emptyset} R_j \right) \right) \right)^+,$$

while if only property (2.15) holds, we can use

$$R_i^+ \cup \left(\left(\left(\bigcup_{\text{ran } R_j \subseteq D} C \circ R_j \right) \cup \left(\bigcup_{\text{ran } R_j \cap \text{dom } R_i = \emptyset} R_j \right) \right)^+ \circ (R_i^+ \cup \text{Id}) \right).$$

It should be noted that if we want the result of the incremental approach to be transitively closed, then we can only apply it if all of the transitive closure operations involved are exact. If, say, the second transitive closure in (2.13) contains extra elements, then the result does not necessarily contain the composition of these extra elements with powers of R_i .

2.5.7 An Omega-like implementation

While the main algorithm of Kelly et al. (1996c) is designed to compute and underapproximation of the transitive closure, the authors mention that they could also compute overapproximations. In this section, we describe our implementation of an algorithm that is based on their ideas. Note that the Omega library computes underapproximations (Kelly et al. 1996b, Section 6.4).

The main tool is Equation (2) of Kelly et al. (1996c). The input relation R is first overapproximated by a “d-form” relation

$$\{\mathbf{i} \rightarrow \mathbf{j} \mid \exists \alpha : \mathbf{L} \leq \mathbf{j} - \mathbf{i} \leq \mathbf{U} \wedge (\forall p : j_p - i_p = M_p \alpha_p)\},$$

where p ranges over the dimensions and \mathbf{L} , \mathbf{U} and \mathbf{M} are constant integer vectors. The elements of \mathbf{U} may be ∞ , meaning that there is no upper bound corresponding to that element, and similarly for \mathbf{L} . Such an overapproximation can be obtained by computing strides, lower and upper bounds on the difference set ΔR . The transitive closure of such a “d-form” relation is

$$\{\mathbf{i} \rightarrow \mathbf{j} \mid \exists \alpha, k : k \geq 1 \wedge k \mathbf{L} \leq \mathbf{j} - \mathbf{i} \leq k \mathbf{U} \wedge (\forall p : j_p - i_p = M_p \alpha_p)\}. \quad (2.16)$$

The domain and range of this transitive closure are then intersected with those of the input relation. This is a special case of the algorithm in Section 2.5.2.

In their algorithm for computing lower bounds, the authors use the above algorithm as a substep on the disjuncts in the relation. At the end, they say

If an upper bound is required, it can be calculated in a manner similar to that of a single conjunct [sic] relation.

Presumably, the authors mean that a “d-form” approximation of the whole input relation should be used. However, the accuracy can be improved by also trying to apply the incremental technique from the same paper, which is explained in more detail in Section 2.5.6. In this case, $C(R_i, D)$ can be obtained by allowing the value zero for k in (2.16), i.e., by computing

$$\{\mathbf{i} \rightarrow \mathbf{j} \mid \exists \alpha, k : k \geq 0 \wedge k \mathbf{L} \leq \mathbf{j} - \mathbf{i} \leq k \mathbf{U} \wedge (\forall p : j_p - i_p = M_p \alpha_p)\}.$$

In our implementation we take as D the simple hull (Section 2.2) of $\text{dom } R \cup \text{ran } R$. To determine whether it is safe to use $C(R_i, D)$, we check the following conditions, as proposed by Kelly et al. (1996c): $C(R_i, D) - R_i^+$ is not a union and for each $j \neq i$ the condition

$$(C(R_i, D) - R_i^+) \circ R_j \circ (C(R_i, D) - R_i^+) = R_j$$

holds.

Bibliography

- Bagnara, R., P. M. Hill, and E. Zaffanella. The Parma Polyhedra Library. <http://www.cs.unipr.it/ppl/>. [154]
- Barthou, D., A. Cohen, and J.-F. Collard (2000). Maximal static expansion. *Int. J. Parallel Program.* 28(3), 213–243. [166]
- Barvinok, A. and K. Woods (2003, April). Short rational generating functions for lattice point problems. *J. Amer. Math. Soc.* 16, 957–979. [147]
- Beletskaya, A., D. Barthou, W. Bielecki, and A. Cohen (2009). Computing the transitive closure of a union of affine integer tuple relations. In *COCOA '09: Proceedings of the 3rd International Conference on Combinatorial Optimization and Applications*, Berlin, Heidelberg, pp. 98–109. Springer-Verlag. [157, 158, 165, 166, 167]
- Boulet, P. and X. Redon (1998). Communication pre-evaluation in HPF. In *EU-ROPAR'98*, Volume 1470 of *Lecture Notes in Computer Science*, pp. 263–272. Springer-Verlag, Berlin. [147]
- Bygde, S. (2010, March). Static WCET analysis based on abstract interpretation and counting of elements. Licentiate thesis. [151, 155]
- Cook, W., T. Rutherford, H. E. Scarf, and D. F. Shallcross (1991, August). An implementation of the generalized basis reduction algorithm for integer programming. Cowles Foundation Discussion Papers 990, Cowles Foundation, Yale University. available at <http://ideas.repec.org/p/cwl/cwldpp/990.html>. [154]
- De Loera, J. A., D. Haws, R. Hemmecke, P. Huggins, and R. Yoshida (2004, January). Three kinds of integer programming algorithms based on barvinok's rational functions. In *Integer Programming and Combinatorial Optimization: 10th International IPCO Conference*, Volume 3064 of *Lecture Notes in Computer Science*, pp. 244–255. [147]
- De Smet, S. (2010, April). personal communication. [160]
- Detlefs, D., G. Nelson, and J. B. Saxe (2005). Simplify: a theorem prover for program checking. *J. ACM* 52(3), 365–473. [147]

- Feautrier, P. (1988). Parametric integer programming. *RAIRO Recherche Opérationnelle* 22(3), 243–268. [147, 148, 152]
- Feautrier, P. (1991). Dataflow analysis of array and scalar references. *International Journal of Parallel Programming* 20(1), 23–53. [147, 149]
- Feautrier, P. (1992, December). Some efficient solutions to the affine scheduling problem. Part II. multidimensional time. *International Journal of Parallel Programming* 21(6), 389–420. [149]
- Feautrier, P., J. Collard, and C. Bastoul (2002). Solving systems of affine (in)equalities. Technical report, PRiSM, Versailles University. [149, 150]
- Galea, F. (2009, November). personal communication. [154]
- Karr, M. (1976). Affine relationships among variables of a program. *Acta Informatica* 6, 133–151. [154]
- Kelly, W., V. Maslov, W. Pugh, E. Rosser, T. Shpeisman, and D. Wonnacott (1996a, November). The Omega calculator and library. Technical report, University of Maryland. [164]
- Kelly, W., V. Maslov, W. Pugh, E. Rosser, T. Shpeisman, and D. Wonnacott (1996b, November). The Omega library. Technical report, University of Maryland. [170]
- Kelly, W., W. Pugh, E. Rosser, and T. Shpeisman (1996c). Transitive closure of infinite graphs and its applications. In C.-H. Huang, P. Sadayappan, U. Banerjee, D. Gelernter, A. Nicolau, and D. A. Padua (Eds.), *Languages and Compilers for Parallel Computing, 8th International Workshop, LCPC'95, Columbus, Ohio, USA, August 10-12, 1995, Proceedings*, Volume 1033 of *Lecture Notes in Computer Science*, pp. 126–140. Springer. [157, 163, 166, 167, 168, 169, 170]
- Meister, B. (2004, December). *Stating and Manipulating Periodicity in the Polytope Model. Applications to Program Analysis and Optimization*. Ph. D. thesis, Université Louis Pasteur. [151]
- Meister, B. and S. Verdoolaege (2008, April). Polynomial approximations in the polytope model: Bringing the power of quasi-polynomials to the masses. In J. Sankaran and T. Vander Aa (Eds.), *Digest of the 6th Workshop on Optimization for DSP and Embedded Systems, ODES-6*. [151]
- Nelson, C. G. (1980). *Techniques for program verification*. Ph. D. thesis, Stanford University, Stanford, CA, USA. [147]
- Schrijver, A. (1986). *Theory of Linear and Integer Programming*. John Wiley & Sons. [160]
- Seghir, R. and V. Loechner (2006, October). Memory optimization by counting points in integer transformations of parametric polytopes. In *Proceedings of the International Conference on Compilers, Architectures, and Synthesis for Embedded Systems, CASES 2006, Seoul, Korea*. [155]

- Tarjan, R. (1972). Depth-first search and linear graph algorithms. *SIAM Journal on Computing* 1(2), 146–160. [164]
- Verdoolaege, S. (2006). `barvinok`, version 0.22. Available from <http://freshmeat.net/projects/barvinok/>. [147]
- Verdoolaege, S. (2009, April). An integer set library for program analysis. Advances in the Theory of Integer Linear Optimization and its Extensions, AMS 2009 Spring Western Section Meeting, San Francisco, California, 25-26 April 2009. [156]
- Verdoolaege, S., K. Beyls, M. Bruynooghe, and F. Catthoor (2005). Experiences with enumeration of integer projections of parametric polytopes. In R. Bodik (Ed.), *Proceedings of 14th International Conference on Compiler Construction, Edinburgh, Scotland*, Volume 3443 of *Lecture Notes in Computer Science*, Berlin, pp. 91–105. Springer-Verlag. [147, 155]
- Verhaegh, W. F. J. (1995). *Multidimensional Periodic Scheduling*. Ph. D. thesis, Technische Universiteit Eindhoven. [155]